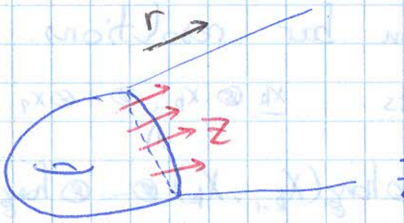
Rem: normally, $HW^*(-,-)$ and $\mathcal{W}(E)$ are defined without reference to $W: E \to \mathbb{C}$ as follows, assuming that E is Liouville:



Z Liouville vector field (near ∞), meaning $d(i_Z \omega) = \omega$.

Z gives a coordinate r on E (near ∞)

Define $HW^*(K,L) = \varinjlim_{\tau > 0} HF^*(K, L; H_\tau)$

OR $= HF^*(K, L; H_{r^2})$

where H_τ is a Hamiltonian which is asymptotically τr near ∞, and H_{r^2} is asymptotically r^2.

01/06/16

Generation criteria for Fukaya categories [Abouzaid] and for $\mathcal{F}(E,W)$ categories [Abouzaid-Ganatra].

We seek criteria under which a given collection of lagrangians $\{L_i\}_{i=1}^{n}$ (split-) generate the entire Fukaya category.

Decategorified analogy: if one wants to show that a given collection of vectors $\{\vec{v}_i\}_{i=1}^{n}$ spans a vector space V, it suffices to show that in $End(V) \leftrightarrows V^* \otimes V$, $id_V = \sum a_{ij} \phi_i^* \otimes v_j$

$\uparrow$ some elements of V^* (ex: v_i^*) if $\exists \langle -,- \rangle$.

Then, for any $w \in V$, $w = id_V(w) = \sum_j \left(\sum_i a_{ij} \phi_i^*(w)\right) v_j$

§1. Hochschild invariants of A_∞-categories.

To a pair (A,B) where A is an associative algebra and B a bimodule over A, get
* Hochschild homology: $HH_*(A,B) = Tor_{A\otimes A^{op}}(A,B) = H^*(A \otimes^L_{A\otimes A^{op}} B)$
* Hochschild cohomology: $HH^*(A,B) = Ext_{A\otimes A^{op}}(A,B) = H^*(Rhom_{A\otimes A^{op}}(A,B))$

Shorthand: $HH_*(A) := HH_*(A,A)$ and $HH^*(A) = HH^*(A,A)$.

$\uparrow$ $(*)$

Given an A_∞-category $\mathcal{C}$, we can directly define a chain complex whose cohomology computes HH^*, HH_*, by adapting the explicit complexes in $(*)$ coming from bar resolutions.

Define

denote elements $\underline{x_k} \otimes x_{k-1} \otimes \dots \otimes x_1$

$$CC_*(\mathcal{C},\mathcal{C}) =: CC_*(\mathcal{C}) := \bigoplus_{\substack{X_0,\dots,X_k \\ \in \mathrm{ob}\,\mathcal{C}}} \hom_{\mathcal{C}}(X_k, X_0) \otimes \hom_{\mathcal{C}}(X_{k-1}, X_k) \otimes \dots \otimes \hom_{\mathcal{C}}(X_0, X_1)$$

"Hochschild chains"

$$CC^*(\mathcal{C},\mathcal{C}) := \prod_{X_0,\dots,X_k} \hom_{vect}\left(\hom_{\mathcal{C}}(X_{k-1}, X_k) \otimes \dots \otimes \hom_{\mathcal{C}}(X_0, X_1),\ \hom_{\mathcal{C}}(X_0, X_k)\right)$$

"Hochschild cochains"

because $\oplus$ inside $\hom_{vect}$, we took it out

The differential involves summing over ways to apply μ's:

For instance,

$$\delta_{CC_*}(\underline{x_k} \otimes \dots \otimes x_1) := \sum (-1)^* \underline{x_k} \otimes \dots \otimes \mu^i(x_{i+j}, \dots, x_{j+1}) \otimes x_j \otimes \dots \otimes x_1$$
$$+ \sum (-1)^* \mu^i(x_s, \dots, x_1, \underline{x_k}, \dots, x_{i+j+1}) \otimes x_{i+j} \otimes \dots \otimes x_{s+1} \quad \text{(cyclic)}$$

$\delta_{CC^*}(\psi) := \mu \circ \hat{\psi} \mp \psi \circ \hat{\mu}$, using our previous notation $\hat{\ }$.

The cohomologies are denoted $HH_*(\mathcal{C})$ and $HH^*(\mathcal{C})$; graded if $\mathcal{C}$ is.

More generally, can take $HH_*(\mathcal{C}, \mathcal{B})$, where $\mathcal{B}$ is an A_∞-bimodule over $\mathcal{C}$, i.e. a bilinear functor $\mathcal{B}: \mathcal{C}^{op} \times \mathcal{C} \to Ch_{\mathbb{k}}$

$$CC_*(\mathcal{C}, \mathcal{B}) = \bigoplus \mathcal{B}(X_k, X_0) \otimes \hom_{\mathcal{C}}(X_{k-1}, X_k) \otimes \dots \otimes \hom_{\mathcal{C}}(X_0, X_1).$$

§2. Open-closed and closed-open maps.

Fix a field $\mathbb{k}$ ex: $\mathbb{k} = \Lambda_{\mathbb{C}}$, $q \in \mathbb{k}$ formal variable

Say X is a compact symplectic manifold (for instance monotone, but take any other setting where all structures are defined).

↳ $\mathcal{F}(X)$ Fukaya category ($\mathbb{Z}$-graded if $2c_1(X) = 0$; otherwise $\mathbb{Z}/2$ or $\mathbb{Z}/2k$-graded).

↳ $QH^*(X)$ quantum cohomology; same grading as above. As a vector space, $QH^*(X) := H^*(X; \mathbb{k})$ (with grading collapsed)