Rem: normally, \( HW^*(\cdot, \cdot) \) and \( W(\cdot) \) are defined without reference to \( W: E \rightarrow C \) as follows, assuming that \( E \) is Liouville:

\[ Z \text{ Liouville vector field (near } \infty) , \]

meaning \( d(i_z \omega) = \omega \).

\( Z \) gives a coordinate \( r \) on \( E \) (near \( \infty \)).

Define \( HW^*(K, L) = \lim_{\varepsilon \to 0} HF^* (K, L; H^r) \)

OR \( = HF^*(K, L; H^r, r) \)

where \( H^r \) is a Hamiltonian which is asymptotically \( T \) near \( \infty \), and \( H^r, r \) is asymptotically \( r^2 \).

**Generation criteria for Fukaya categories [Abozad]**

**and for \( \mathbb{F}(E, W) \) categories [Abozad - Sanaa].**

We seek criteria under which a given collection of lagrangians \( \{ L^j \}, j = 1, \ldots \) (split) generate the entire Fukaya category.

Decategorified analogy: if one wants to show that a given collection of vectors \( \{ v_j \}, j = 1, \ldots \) span a vector space \( V \), it suffices to show that in \( \text{End}(V) \) \( \mathbb{Z} = V^* \otimes V \), i.e.,

\[ \text{id}_V = \sum a_j \, \phi^*_j \otimes v_j \]

Then, for any \( v \in V \), \( v = \text{id}_V(v) = \sum a_j (\phi_j^*(w) \otimes v) \).

**§1. Hochschild invariants of \( A_\infty \)-categories**

To a pair \((A, B)\) where \( A \) is an associative algebra and \( B \) a bimodule over \( A \), get \( \ast \) Hochschild homology: \( HH^\ast(A, B) = \text{Ext}_{A_\infty^B}(A, B) = H^\ast(A \otimes_{A_\infty^B} B) \)

\( \ast \) Hochschild cohomology: \( HH_\ast(A, B) = \text{Ext}_{A_\infty^B}(B, A) = H^\ast(Rhom_{A_\infty^B}(A, B)) \)

Shorthand: \( HH^\ast(A) = AH^\ast(A, A) \) and \( HH_\ast(A) = H^\ast(A, A) \).
Given an $A_{oo}$ category $C$, we can directly define a chain complex whose cohomology computes $HH^*, HH_*$, by adopting the explicit complex in (*)& coming from bar resolutions.

Define $\bigoplus_{X_0 \cdots X_n \in C} \text{Hom}(X_0, X_n) \otimes \cdots \otimes \text{Hom}(X_{n-1}, X_0)$.

$CC^*(C, C) := \bigoplus_{X_0 \cdots X_n \in C} \text{Hom}(X_0, X_n) \otimes \cdots \otimes \text{Hom}(X_{n-1}, X_0)$

"Hochschild cocycles" because $\otimes$ inside Hom;

we think of it as

The differential involves summing over ways to apply $\psi$'s.

For instance,

$\delta_{CC^*}(X_0 \otimes \cdots \otimes X_n) := \sum (-1)^* x_0 \otimes \cdots \otimes x_j \cdot \cdots \otimes x_i \otimes \cdots \otimes x_n \cdot \psi \cdot \cdots \cdot \psi \cdot \cdots \cdot \psi$

(duckbill)

$\delta_{CC^*}(\psi) := \rho \circ \delta \circ \psi \circ \rho$, using our previous notation $\wedge$.

The cohomologies are denoted $HH^*_C(C)$ and $HH^*_C(C)$; graded if $C$ is.

More generally, can take $HH^*_B(C, B)$, where $B$ is an $A_{oo}$-module over $C$, i.e. a bilinear functor $B : C \otimes C \to C$.

$CC^*_C(C, B) := \bigoplus_{X_0 \cdots X_n} \text{Hom}(X_0, X_n) \otimes \cdots \otimes \text{Hom}(X_{n-1}, X_0) \otimes \cdots \otimes \text{Hom}(X_n, X_0)$.

$\S2.$ Open-closed and closed-open maps.

Fix a field $k$, $q \in k$.

Say $X$ is a compact symplectic manifold (for instance a torus, but take any other setting where all structures are defined).

1. $F(X)$ Fukaya category ($\mathbb{Z}$-graded if $2c_1(X) = 0$; otherwise $\mathbb{Z}/2$ or $\mathbb{Z}/2k$-graded).

2. $QH^*(X)$ quantum cohomology; same grading as above. As a vector space, $QH^*(X) := H^*(X; k)$ (with grading collapsed).
\((-,-)_x : \mathbb{Q} \mathcal{H}^*(x)^{\otimes 2} \to k \quad (\alpha, \beta)_x = \int_x \alpha \bullet \beta \)
and w.r.t. \((-,-)_x, \alpha : (\mathbb{Q} \mathcal{H}^*(x))^{\otimes 2} \to \mathbb{Q} \mathcal{H}^*(x)\), equivalent to the
data of "3-point functions"
\((x, \beta, y)_x = (x \bullet \beta, y)_x\), counting \(\bigoplus \bigcup_{(x, y) \in \mathcal{X}} \text{PD}(x)\)
weighted by \(q^{\langle x, y \rangle}\).

[Seidel, \textit{CFO}, Aboveaid]: there are geometric maps
\(OE : \text{HH}_{x-n} (F(x)) \to \mathbb{Q} \mathcal{H}^*(x)\)
\(\&_O : \mathbb{Q} \mathcal{H}^*(x) \to \text{HH}^*(F(x))\).

**Proposition:** \(OE\) is a ring map
\(OE\) is a \(\mathbb{Q} \mathcal{H}^*(x)\)-module map, where the \(\mathbb{Q} \mathcal{H}^*\)
module structure on \(\text{HH}_x\) is induced by \(OE\) and the \(\text{HH}^*\)-module structure on
\(\text{HH}_x\) (non-commutative).

It is broadly expected that \((OE, \&_O)\) the "Hochschild calculus"
with standard operations on \(\mathbb{Q} \mathcal{H}^*(x)\).

How do define these maps, broadly?

* Given \(x_0 \circ x_1 \circ \ldots \circ x_n\), to define \(OE\), it suffices to specify:
\((OE(x_0 \circ x_1 \circ \ldots \circ x_n), \beta)_x = \bigoplus_{x_1, \ldots, x_n} \text{PD}(x)\)
weighted count by \(q^{\langle x, y \rangle}\).

* Given \(\beta \in \mathbb{Q} \mathcal{H}^*(x)\),
\(\&_O(\beta)(x_0, \ldots, x_n) = \sum_{x_0} \bigoplus_x \text{PD}(p)\).

We needed to choose cycles, but the result is independent of choices.

**Proposition:** these chain level \(OE\) and \(\&_O\) descend to cohomology; call the
cohomology level maps \(OE\) and \(\&_O\) too.

**Proof:** analyze cochain a breaking.
Rem. when \( X \) is monotone, both \( \mathcal{O}H^+(x) \) and \( F(x) \) decompose into summands indexed by \( \nu \) where \( \nu \) is an eigenvalue of \( (c_\nu(x), x) \). \( \mathcal{O}H^+(x) \rightarrow \mathcal{O}H^+(x) \).

Call \( \mathcal{O}H^+(x)_\nu \), \( F_\nu(x) \) the corresponding summands.

**Proposition:** [Ritter-Smith; see also Sheridan] The maps \( \mathcal{O}F \) and \( \mathcal{O}F \) respect these decompositions.

**Proposition:** \( \mathcal{O}F \) and \( \mathcal{O}F \) are "linear dual" in the following sense. Have \((\cdot, \cdot)_x: \mathcal{O}H^+(x) \cong \mathcal{O}H^+(x)^v\), and \( F(x) \) (when \( X \) is compact, or rather when \( L \)'s are) is a "weak C-Y category" (some refinement of \( HF^+(K, L) \cong HF^+(L, K)^v \), which implies \( HH^+_x(F(x))^v \cong HH^+(F(x)) \)). We have:

\[
\begin{array}{ccc}
\mathcal{O}H^+ & \xrightarrow{\mathcal{O}F} & HH^+_x(F(x)) \\
\mathcal{O}H^+(x)^v \downarrow & & \downarrow CY^+ \\
& \mathcal{O}F^v & HH^+_x(F(x))^v
\end{array}
\]

### §3: Abouzaid's Generation Criterion:

**Theorem [Abouzaid]** say \( \mathcal{A} \subset F(x) \), full subcategory, have:

\[ \mathcal{O} \mathcal{A} \mathcal{A} : HH^+_{x,n}(\mathcal{A}, \mathcal{A}) \rightarrow HH^+_{x,n}(F, F) \rightarrow \mathcal{O}H^+(x) \]

If \( \mathcal{O} \mathcal{A} \mathcal{A} \) hits \( \mathcal{A} \subset \mathcal{O}H^+(x) \), then \( \mathcal{A} \) split-generates \( F(x) \).

(Originally for wrapped Fukaya categories, implemented for compact Fukaya categories by Abouzaid; FOOO, Ritter-Smith, Sheridan, Pantev-Sheridan.)

If \( \mathcal{A} \subset F_{\nu}(x) \) and \( \mathcal{O} \mathcal{A} \mathcal{A} \) hits \( \mathcal{P}_{\mathcal{O}H^+(x)_\nu}(x) \), then \( \mathcal{A} \) split-generates \( F_{\nu}(x) \).

"Works one summand at a time."
Sketch of proof: "annulus argument" (or Cartan condition)

Baby case: note that there is a map \( HF^*(L,L) \rightarrow HH^*_x(A) \) for any \( L \in \text{ob} A \) (\( \text{Hom}_A(L,L) \) subcomplex of \( CC_*(A) \)).

There is also a map \( HH^*_x(A,A) \rightarrow HF^*(k,k) \) for any \( k \in \text{ob} A \) (\( \text{Hom}_A(k,k) \) quotient complex of \( CC_*(A,A) \)).

Suppose \( \partial \in \text{ob} A \): \( HF^*(L,L) \rightarrow HH^*_x(A,A) \) hits 1.

\[ \xymatrix{ L \ar^\partial @<1ex>[r] & X \ar^-Q @<1ex>[r] & QHF^*(x) \ar^-QH^*(x) } \]

For an arbitrary \( K \in F(x) \), have \( \xi_0^L: QH^*(x) \rightarrow HF^*(K,K) \).

Claim: the map \( n \mapsto 1 \) always

(exercise)

Claim: if \( L, K, \exists \) a comm. diagram

\[ \xymatrix{ HF^*(L,L) \ar[d]^{\xi_0^L} \ar[r]^{\Delta} & HF^*(L,K) \otimes HF^*(K,L) \ar[d]^{[\mu]} \ar[r] & HF^*(K,K) \ar[d]^{\Delta^*} \ar[l]^{\xi_0^K} } \]

\( \Delta^* \) is "cooperad" a "new operation".

So, if \( \xi_0^L \) hits 1, then for any \( K \),

\[ HF^*(L,K) \otimes HF^*(K,L) \xrightarrow{\mu} HF^*(K,K) \]

hits \( 1_K \).

meaning that in \( H^F(x) \),

\[ K \xrightarrow{\xi_0^L} L \xrightarrow{\xi_0^K} K \]

so

any \( K \) is a summand of \( L \).

Proof of 2nd claim: \[ \xymatrix{ 0 \ar[r]^{\xi_0^L} \ar@{=}[d] & 0 \ar[r]^{\xi_0^K} \ar@{=}[d] & 0 \ar[r]^{\xi_0^K} \ar@{=}[d] & 0 \ar[r]^{\xi_0^K} \ar@{=}[d] & 0 \ar[r]^{\xi_0^K} } \]

\( \xi_0^L \), \( \xi_0^K \) count, because
\[ O \varepsilon (x) = \sum_{\beta \in \text{CH}(x) \text{ basis}} O \varepsilon \beta_i (x) \cdot \beta_i \]

\[ \varepsilon O (\beta_i) = \sum_{\beta \in \text{CH}(x) \text{ basis}} \ldots \]

\[ O \varepsilon \cdot O \varepsilon (x) = \sum_{\beta \in \text{CH}(x) \text{ basis}} O \varepsilon \beta_i (x) \cdot O \varepsilon (\beta_i) \quad (\text{as } f \cdot f (\beta) = \sum \beta_i \cdot f (\beta_i)) \]

\[ = \sum (O \varepsilon, O \varepsilon) \beta_i \cdot \beta_i \]

\[ = (O \varepsilon, O \varepsilon)_A \quad \text{(Homologically)} \]

And \[ \begin{array}{c}
\text{More general case: there's a commutative diagram (Above said)}:
\end{array} \]

\[ \begin{array}{ccc}
\text{HH}_p (A, A) & \longrightarrow & H^p (Y^r \otimes_A Y^l) \\
\downarrow O \varepsilon & & \downarrow \varepsilon P \\
\text{QH}^*(X) & \xrightarrow{O \varepsilon} & \text{HP}^*(K, K) \\
\end{array} \]

where \[ Y^r \otimes_A Y^l \] is a chain complex of the form

\[ \bigoplus \text{hom}_A (X_k, K) \oplus \text{hom}_A (X_{k-1}, X_k) \oplus \ldots \oplus \text{hom}_A (X_0, X_1) \oplus \text{hom}_A (K, K_0) \]

This diagram implies that if \( O \varepsilon \) hits \( 1 \), then \( \varepsilon P \) hits \( 1 \) for any \( K \).

\( (i): \lambda \in F (A) \)

\( \Rightarrow \text{hom} (K, \varepsilon P \otimes X) \overset{\lambda}{\longrightarrow} \text{hom} (K, K) \) hits \( 1 \)

\( \Rightarrow A = 0 \) in \( H^*(\text{hom}_{\text{F/A}} (K, K)) \)

\( \Rightarrow K = 0 \) in \( F/A \)

\( \Rightarrow K \) is split-generated by \( A \).

[Theorem: If \( O \varepsilon \) hits \( 1 \), then \( O \varepsilon \) and \( \varepsilon P \) are isomorphisms.]
There are many instances in which one can verify this generation criterion. By PD at $O^\mathbb{F}$ and $O$, it suffices to show $\overline{c_0} : \text{QH}^*(X) \to \text{HH}^*(A)$ is injective: it implies $O^\mathbb{F}$ is surjective, hence hits 1.

There are many cases in which a given $\text{QH}^*(X)$ is rank 1: $\leq \operatorname{rk}_x \left( \frac{\text{pr}^\mathbb{F}}{\text{pr}^x} \right)$

In this case, if $A \leq F^w(x)$, and $A$ has any $L$ with $\text{HF}^*(L,L) \neq 0$, $A$ satisfies the generation criterion ("semi-simple case")

Indeed, $\overline{c_0} : \text{QH}^*(X) \to \text{HH}^*(A) \to \text{HF}^*(L,L)$

ex: $\mathbb{P}^1$, or more generally, $\mathbb{P}^n$ (QH*(X) splits into rank 1 summands)

The Clifford torus with all its almost systems generate, for instance.

There are other cases in which one can deduce a "automatic generation":

**Theorem [Ganatra] Say:**

1. $A$ is "homologically smooth" (some condition only depending on $A$

2. $\operatorname{rk} \text{HH}^0(A) \geq \operatorname{rk} \text{QH}^*(X)_x \operatorname{rk} \text{if} A \leq F(x)$

Then, $A$ split-generates.

Can apply this to other cases, such as Fano varieties, Fano hypersurfaces in $\mathbb{P}^n$, using computations of $[\text{Ch}^n, \text{Ch}^n, A^\text{fano}]$ and [Smith; Shen].
Returning to LG models $(E, W)$, there is a map
$$\text{HH}_*(F(E, W), B_{2w}) \rightarrow \text{HF}^*(E, W)$$

[Theorem (Abadeh-Santra)] if $O_E W/\mathbb{A}$ hits 1, it split-generates.

Expectation: $O_E W$ is always an isomorphism, at least when $W$
is a Lefschetz fibration (true with a critical point,...)