Math 535a Homework 1

Due Wednesday, January 25, 2017 by 5 pm

Please remember to write down your name on your assignment.

1. Show that the induced topology (for a subset $X \subset Y$ of a topological space Y) and the quotient topology (for a surjection $X \twoheadrightarrow Y$ from a topological space X onto a set Y) satisfy the axioms of a topological space.

2. Show that the topological spaces $S^1 \subset \mathbb{R}^2$ (with topology induced by the inclusion into $\mathbb{R}^2$ and $[0,1]/\{0,1\}$ (with the quotient topology from the topology on $[0,1] \subset \mathbb{R}$) are homeomorphic.

3. Prove that S^1, with either topology considered above, is a topological manifold.

4. Show that the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$, if it exists at a point $a \in \mathbb{R}^n$, is unique.

5. Produce, with proofs, examples of the following topological spaces which are not topological manifolds:
   a) A space X which is locally Euclidean[1] and second countable, but not Hausdorff.

   b) A space X which is Hausdorff and second countable, but not locally Euclidean.

6. Let $S^n = \{(x_1, \ldots, x_{n+1}) | x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$. Prove that S^n has the structure of a smooth manifold, using charts associated to the cover $U_N = \{x_1 \neq +1\}$, $U_S = \{x_1 \neq -1\}$. (Hint: as in the case of S^1 in class, use *sterographic projection* to map U_N, respectively U_S to $\mathbb{R}^n$).

7. Prove that the antipodal map $S^n \to S^n$, $\mathbf{x} \mapsto -\mathbf{x}$ is a diffeomorphism of manifolds.

8. Let h be a continuous real-valued function on $S^1 = \{x^2 + y^2 = 1 \subset \mathbb{R}^2\}$ satisfying $h(0,1) = h(1,0) = 0$ and $h(-x_1, -x_2) = -h(x_1, x_2)$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} ||x|| h(x/||x||) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   a) Show that f is continuous at $(0,0)$, that the partial derivatives of f at $(0,0)$ are defined, and that more generally all directional derivatives of f are defined.

   b) Show that f is not differentiable at $(0,0)$ except if h is identically zero.

9. Finish the proof from class that $\mathbb{R}P^n$ is a smooth manifold (of dimension n)

[1] in the sense that each point $p \in X$ has a neighborhood homeomorphic to an open set of $\mathbb{R}^n$.

10. Finish the proof from class that $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a smooth 2-manifold.