Homework 1

Exercise 1. Show that the induced topology (for a subset $X \subset Y$ of a topological space Y) and the quotient topology (for a surjection $X \twoheadrightarrow Y$ from a topological space X onto a set Y) satisfy the axioms of a topological space.
Solution. Since $\emptyset, Y \subset Y$ are open in Y, the intersections $\emptyset = \emptyset \cap X$ and $X = Y \cap X$ are open in X. Given a family (U_α) of open sets in X write $U_\alpha = V_\alpha \cap X$ for open sets $V_\alpha \subset Y$ and observe that

$$\bigcup_\alpha U_\alpha = \bigcup_\alpha (V_\alpha \cap X) = X \cap \bigcup_\alpha V_\alpha$$

is open in X since $\bigcup_\alpha V_\alpha$ is open in Y. Similarly, given a finite family (U_i) of open sets in X write $U_i = V_i \cap X$ for open sets $V_i \subset Y$. Then

$$\bigcap_i U_i = \bigcap_i (V_i \cap X) = X \cap \bigcap_i V_i$$

is open in X since $\bigcap_i V_i$ is open in Y.

Let $\pi\colon X \twoheadrightarrow Y$ be surjective. We have $\pi^{-1}(\emptyset) = \emptyset$ and $\pi^{-1}(Y) = X$ and so $\emptyset, Y \subset Y$ are open in the quotient topology. Let (U_α) be a family of open sets in Y. Then

$$\pi^{-1}\big(\bigcup_\alpha U_\alpha\big) = \bigcup_\alpha \pi^{-1}(U_\alpha)$$

is open in X and therefore $\bigcup_\alpha U_\alpha$ is open in Y. Similarly, given a finite family (U_i) of open sets in Y the preimage

$$\pi^{-1}\big(\bigcap_i U_i\big) = \bigcap_i \pi^{-1}(U_i)$$

is open in X. Hence, $\bigcap_i U_i$ is open in Y.

Exercise 2. Show that the topological spaces $S^1 \subset \mathbb{R}^2$ (with topology induced by the inclusion into $\mathbb{R}^2$) and $[0,1]/\{0,1\}$ (with the quotient topology from the topology on $[0,1] \subset \mathbb{R}$) are homeomorphic.
Solution. Consider the continuous function $f\colon [0,1] \longrightarrow S^1$ defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Because $f(0) = f(1)$ this descends to a continous function $g\colon [0,1]/\{0,1\} \longrightarrow S^1$ which is bijective. Furthermore, $[0,1]/\{0,1\}$, being the continuous image of $[0,1]$, is compact and S^1 is Hausdorff because $\mathbb{R}^2$ is. It follows that g is a homeomorphism.

Exercise 3. Prove that S^1, with either topology considered above, is a topological manifold.
Solution. As a subspace of $\mathbb{R}^2$ the circle S^1 is Hausdorff and second countable. It remains to show that it is locally Euclidean. For this it suffices to show that $[0,1]/\{0,1\}$ is locally Euclidean. Let $p \in [0,1]/\{0,1\}$. If $p = [q]$ for $q \in (0,1)$ then the projection $(0,1) \longrightarrow (0,1)/\{0,1\} \subset [0,1]/\{0,1\}$ is a homeomorphism. If $p = [0] = [1]$, observe that the map $\phi\colon [0,1]/\{0,1\} \longrightarrow [0,1]/\{0,1\}$ with $\varphi([x]) = [x + 1/2 \mod 1]$ is continuous and $\varphi(\varphi([x])) = [x]$. Hence ϕ is a homeomorphism and $\phi(p) = [1/2]$. Since $[0,1]/\{0,1\}$ is locally Euclidean at $[1/2]$ this implies that it is also locally Euclidean at p.

Exercise 4. Show that the derivative of a function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, if it exists at a point $a \in \mathbb{R}^n$, is unique.
Solution. Assume that L_1 and L_2 are both derivatives of f at a. This means that

$$f(a+h) = f(a) + L_1(h) + o(\|h\|)$$

and

$$f(a+h) = f(a) + L_2(h) + o(\|h\|)$$

as $h \to 0$. Let $v \in \mathbb{R}^n$ with $\|v\| = 1$ and let $\varepsilon > 0$. Then

$$L_1(v) - L_2(v) = \frac{f(a + \varepsilon v) - f(a) - (f(a + \varepsilon v) - f(a)) + o(\varepsilon)}{\varepsilon} = o(1)$$

as $\varepsilon \to 0$, that is $L_1(v) = L_2(v)$. If $\|v\| \neq 1$ and $v \neq 0$ then this implies

$$L_1(v) = \|v\| L_1(v/\|v\|) = \|v\| L_2(v/\|v\|) = L_2(v).$$

It follows that $L_1 = L_2$.

Exercise 5. Produce, with proofs, examples of the following topological spaces which are not topological manifolds:

(i) A space X which is locally Euclidean and second countable, but not Hausdorff.

(ii) A space X which is Hausdorff and second countable, but not locally Euclidean.

Solution. The standard example of a space X which is locally Euclidean and second countable but not Hausdorff is the "line with two origins". It is obtained by identifying two copies of $\mathbb{R}$ along $\mathbb{R} \setminus \{0\}$. Let 0 and $0'$ denote the two origins in X. The quotient map $\mathbb{R} \sqcup \mathbb{R} \twoheadrightarrow X$ restricted to each of the two copies of $\mathbb{R}$ is a topological embedding. Therefore X is second countable and locally Euclidean. However, any open neighbourhood of 0 intersects any open neighbourhood of $0'$ in X. Hence, X is not Haudorff.

For (ii), let $X = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \subset \mathbb{R}^2$ be the union of the coordinate axes in $\mathbb{R}^2$. Then X is Hausdorff and second countable. If X were a topological manifold, its dimension would have to be 1 by Brouwer's invariance of domain. In particular, given any small enough open ball U centered at 0 in $\mathbb{R}^2$, the intersection $U \cap X$ would have to be homeomorphic to an open interval $I \subset \mathbb{R}$. But removing any single point from I results in a space with two path components while removing the origin from $U \cap X$ results in a space with four path components. This contradiction shows that X cannot be locally Euclidean around 0.

Exercise 6. Let $S^n = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$. Prove that S^n has the structure of a smooth manifold, using charts associated to the cover $U_N = \{x_1 \neq +1\}$, $U_S = \{x_1 \neq -1\}$.

Solution. Take the stereographic projection

$$\varphi_N \colon U_N \longrightarrow \mathbb{R}^n, (x_1, \ldots, x_{n+1}) \longmapsto \frac{1}{1 - x_1}(x_2, \ldots, x_{n+1})$$

with inverse

$$\mathbb{R}^n \longrightarrow U_N, x = (x_1, \ldots, x_n) \longmapsto \frac{1}{\|x\|^2 + 1}(\|x\|^2 - 1, 2x_1, \ldots, 2x_n).$$

Both of these are continuous so we obtain a chart (U_N, φ_N). Similarly,

$$\varphi_S \colon U_S \longrightarrow \mathbb{R}^n, (x_1, \ldots, x_{n+1}) \longmapsto \frac{1}{1 + x_1}(x_2, \ldots, x_{n+1})$$

with inverse

$$\mathbb{R}^n \longrightarrow U_S, x = (x_1, \ldots, x_n) \longmapsto \frac{1}{1 + \|x\|^2}(1 - \|x\|^2, 2x_1, \ldots, 2x_n)$$

defines a chart (U_S, φ_S). We compute the transition maps as

$$\varphi_N(\varphi_S^{-1}(x)) = \frac{1}{\|x\|^2} \cdot x = \varphi_S(\varphi_N^{-1}(x)), \quad \text{for } x \neq 0.$$

Hence, all transition maps are smooth and we have defined a smooth atlas on S^n.

Exercise 7. Prove that the antipodal map $f \colon S^n \longrightarrow S^n, x \longmapsto -x$ is a diffeomorphism of manifolds.

Solution. Since $f^2 = \mathrm{id}$ we only need to check that f is smooth. Let $p = \varphi_N^{-1}(x) \in U_N$ (in the notation of the previous exercise). Then $f(p) \in U_S$ and

$$\varphi_S(f(\varphi_N^{-1}(x))) = \frac{1}{1 + \frac{1-\|x\|^2}{1+\|x\|^2}} \frac{1}{1+\|x\|^2}(-2x_1, \ldots, -2x_n) = -x$$

is a smooth function of $x \in \mathbb{R}^n$. The case $p \in U_S$ is entirely analogous. Hence, f is a smooth function $S^n \longrightarrow S^n$.

EXERCISE 8. Let h be a continuous real-valued function on $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ satisfying $h(0,1) = h(1,0) = 0$ and $h(-x_1, -x_2) = -h(x_1, x_2)$. Define a function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \|x\| \cdot h(x/\|x\|) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(i) Show that f is continuous at $(0,0)$, that the partial derivatives of f at $(0,0)$ are defined, and that more generally all directional derivatives of f are defined.
(ii) Show that f is not differentiable at $(0,0)$ except if h is identically zero.

Solution. Since S^1 is compact and h is continuous, there is some $C \in \mathbb{R}$ such that $|h(x)| \leq C$ for all $x \in S^1$. Hence,

$$|f(x)| \leq C\|x\| \longrightarrow 0$$

as $x \to 0$ and therefore f is continuous at 0. Given $v \in \mathbb{R}^2 \smallsetminus \{0\}$ we compute the directional derivative of f at 0 in the direction of v as

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon v)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{|\varepsilon| \|v\| h(\varepsilon v/\|\varepsilon v\|)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\varepsilon \|v\| h(v/\|v\|)}{\varepsilon} = \|v\| h(v/\|v\|).$$

In particular, all directional derivatives of f exist at 0.

However, assuming that f is differentiable at 0, by the chain rule we would have

$$h(v) = \partial_v f(0) = v_1 \partial_1 f(0) + v_2 \partial_2 f(0) = 0$$

for all $v \in S^1$.

EXERCISE 9. Finish the proof from class that $\mathbb{RP}^n$ is a smooth manifold of dimension n.

Solution. In homogeneous coordinates let $U_i = \{[x_0 : \cdots : x_n] : x_i = 1\} \subset \mathbb{RP}^n$. Then the map $\varphi_i\colon U_i \longrightarrow \mathbb{R}^n$ with $\varphi_i([x_0 : \cdots : x_n]) = (x_0/x_i, \ldots, x_n/x_i)$ is a homeomorphism. Since $\mathbb{RP}^n$ is covered by a finite number of charts it is automatically second countable. To check that it is Hausdorff, observe that using a linear automorphism of $\mathbb{RP}^n$ we can move any two points of $\mathbb{RP}^n$ to $[1 : 0 : 0 : \cdots : 0]$ and $[1 : 1 : 0 : \cdots : 0]$. Both of these lie in U_0 and therefore can be separated by open sets. Pulling these open sets back along the automorphism shows that $\mathbb{RP}^n$ is Hausdorff.

Finally, we need to look at the transition maps to check that $\mathbb{RP}^n$ is smooth. We compute

$$\varphi_i(\varphi_j^{-1}(x_0, \ldots, \widehat{x_j}, \ldots, x_n)) = \varphi_i([x_0 : \cdots : x_{j-1} : 1 : x_{j+1} : \cdots : x_n]) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{j-1}}{x_i}, \frac{1}{x_i}, \frac{x_{j+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

which is smooth for $x_i \neq 0$.

EXERCISE 10. Finish the proof from class that $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a smooth 2-manifold.

Solution. The quotient map $\mathbb{R}^2 \longrightarrow\!\!\!\!\rightarrow \mathbb{R}^2/\mathbb{Z}^2$ is open. Therefore, $\mathbb{T}^2$ is second countable. Furthermore, given any two distinct points $x, y \in \mathbb{T}^2$ we can choose $p, q \in \mathbb{R}^2$ such that $[p] = x$, $[q] = y$ and p and q lie in the interior of a rectangle U with sidelength 1. Then the quotient map $\mathbb{R}^2 \longrightarrow\!\!\!\!\rightarrow \mathbb{R}^2/\mathbb{Z}^2$ restricted to U is a topological embedding and we can separate p and q by open sets in U. The images of these open sets will then separate x and y in $\mathbb{T}^2$.

To see that $\mathbb{T}^2$ is a smooth manifold, let $x = [p] \in \mathbb{T}^2$ and consider $U_x = p + (-1/2, 1/2)^2 \subset \mathbb{R}^2$. Since U_x is an open rectangle with sidelength 1, the restriction of $f\colon \mathbb{R}^2 \longrightarrow\!\!\!\!\rightarrow \mathbb{T}^2$ to U_x is a topological embedding. Therefore, there is a homeomorphism $\varphi_x\colon f(U_x) \longrightarrow U_x$ and the collection $\{(f(U_x), \varphi_x)\}_{x \in \mathbb{T}^2}$ is an atlas for $\mathbb{T}^2$. The transition maps are just the identity, hence this is a smooth atlas.