Math 535a Homework 2

Due Friday, February 3, 2017 by 5 pm

Please remember to write down your name on your assignment.

1. Show that the two definitions of a submanifold $Y^m \subset N^n$ given in class are equivalent. Namely, show that Y is the image of an embedding $M^m \hookrightarrow N^n$ if and only if at every point $p \in Y$, there exists a chart (U, ϕ) in N's maximal atlas, containing (and centered at) p, such that $\phi(U \cap Y) = \phi(U) \cap \{x_{m+1} = x_{m+2} = \cdots = x_n = 0\} = \phi(U) \cap (\mathbb{R}^m \times \{0\})$.

2. Prove the following result: if $f : M^m \to N^n$ is a submersion between two smooth manifolds, or more generally if f is simply a smooth map and $y \in N$ is a regular value of f, then $S := f^{-1}(y)$ has the structure of a smooth submanifold of M of dimension $m - n$.

   Remarks:
   - You are welcome to use (and probably should use) the implicit function theorem.
   - The result you are being asked to prove is slightly stronger than the result stated as a "Corollary" in class. Namely, you are being asked to prove not just that S can be given the structure of a smooth manifold, but in fact that S with the smooth manifold structure that it can be given is naturally a submanifold of N. You may use either definition of *submanifold*, as you will prove below that both definitions are equivalent.
   - **Hint**: In class, we sketched the construction of a chart containing any point $p \in S$. You are welcome to recall, with detail, this construction, and may want to analyze it to produce a chart (U, ϕ) of a neighborhood of p in N, whose intersection with S maps to the intersection of $\phi(U)$ with $\mathbb{R}^m \times \{0\}$. Finally, you must prove that transition functions between any two such charts are C^∞).

3. Prove that $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ can be given the structure of an n-dimensional manifold by exhibiting it as the regular value of some map.

4. Let $M \subset \mathbb{R}^N$ be a submanifold. In class, we gave a first definition of the tangent space to M at a point p as follows: a vector $\vec{v} \in \mathbb{R}^N$ is said to be tangent to M at p if there exists a smooth parametrized curve $\alpha : (-\epsilon, \epsilon) \mapsto \mathbb{R}^N$ with $im(\alpha) \subset M$, $\alpha(0) = p$, and $\alpha'(0) = \vec{v}$. The *tangent space* $T_pM \subset \mathbb{R}^N$ is then the set of all tangent vectors to M at p.

   Prove that T_pM is a vector space (or equivalently, that $T_pM \subset \mathbb{R}^N$ is a linear subspace).

5. Let $O(n) = \{A \in M_n(\mathbb{R}) | AA^T = I\}$ be the *orthogonal group*, where A^T is the *transpose* of A. Consider the map

$$\phi : M_n(\mathbb{R}) \to \mathrm{Sym}(n)$$
$$A \mapsto AA^T$$

   where $Sym(n) = \{B \in M_n(\mathbb{R}) | B = B^T\}$ is the set of *symmetric matrices*.

(a) Show that $\mathrm{Sym}(n)$ is a submanifold of $M_n(\mathbb{R})$ (and in particular a manifold), and compute its dimension. (**Hint**: It may be helpful to first prove, then apply, the following general Lemma: If V is a finite-dimensional vector space, it naturally has the structure of a smooth manifold, and if $W \subset V$ is a linear subspace, then it is naturally a submanifold of V).

(b) Prove that $I \in \mathrm{Sym}(n)$ is a regular value of ϕ.

(c) Prove that $O(n)$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension?

(d) Prove that $O(n)$ is compact.

6. Let Γ be a group and M a smooth manifold. A (C^∞) *action* of Γ on M is a group homorphism ρ from Γ to the group $\mathrm{Diff}(M)$ of diffeomorphisms on M. If $\gamma \in \Gamma$ and $x \in M$, we write $\gamma x = \rho(\gamma)(x)$ for the image of x under the diffeomorphism $\rho(\gamma)$.

Recall from class that the *quotient space* M/Γ of the action Γ on M is the set of equivalence classes of the equivalence relation $\sim$ defined by $x \sim y$ iff $y = \gamma x$ for some $\gamma \in \Gamma$.

(a) We say the action of Γ on M is *discontinuous* if, for every compact subset K of M, the set $\{\gamma \in \Gamma | K \cap \gamma K \neq \emptyset\}$ is finite. We say the action of Γ on M is *free* if $\gamma x \neq x$ for every $x \in M$ and $\gamma \in \Gamma - \{\mathrm{id}\}$.

Prove that if Γ acts freely and discontinously on M, then the quotient M/Γ naturally has the structure of a smooth manifold.

(b) Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ act on $S^n \subset \mathbb{R}^{n+1}$ by sending $x \mapsto -x$. Using the standard manifold structure on S^n (either as given above via expressing S^n as a preimage or as studied on homework last week), prove that $S^n/\mathbb{Z}_2$ has the structure of a manifold, which is diffeomorphic to $\mathbb{R}P^n$, equipped with the smooth manifold structure which you defined on your homework last week: (with charts $U_i = \{x_i \neq 0\}, \phi_i : U_i \mapsto \mathbb{R}^n$, $[x_0 : \cdots x_n] \mapsto (\frac{x_0}{x_i}, \frac{x_1}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i})$.