Homework 2

Exercise 2.1. Show that the two definitions of a submanifold $Y^m \subset N^n$ given in class are equivalent. Namely, show that Y is the image of an embedding $M^m \hookrightarrow N^n$ if and only if at every point $p \in Y$ there exists a chart (U, ϕ) in N's maximal atlas, containing (and centered at) p, such that

$$\phi(U \cap Y) = \phi(U) \cap \{x_{m+1} = \cdots = x_n = 0\} = \phi(U) \cap (\mathbb{R}^m \times \{0\}).$$

Solution. Assume first that $f\colon M^m \hookrightarrow N^n$ is an embedding and let $p \in M^m$. Choose a chart (U, φ) in M^m around p with $\varphi\colon U \longrightarrow \mathbb{R}^m$ a diffeomorphism and $\varphi(p) = 0$. Since f is a homeomorphism onto its image there is some open set $V \subset N^n$ such that $f(U) = f(M) \cap V$ and by shrinking U and V we can assume that there is a diffeomorphism $\psi\colon V \longrightarrow \mathbb{R}^n$ with $\psi(f(p)) = 0$. We obtain a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f|_U} & V \\ \varphi\downarrow & & \downarrow\psi \\ \mathbb{R}^n & \xrightarrow[g]{} & \mathbb{R}^m \end{array}$$

for a uniquely determined embedding g. By the implicit function theorem there is a neighbourhood V_0 of $0 \in \mathbb{R}^m$ and a diffeomorphism $F\colon V_0 \longrightarrow W \subset \mathbb{R}^n$ such that setting $U_0 = g^{-1}(V_0)$ we have

$$F(g(x_1, \ldots, x_m)) = (x_1, \ldots, x_m, 0 \ldots, 0), \quad \text{for } (x_1, \ldots, x_m) \in U_0.$$

Set $\widetilde{U} = \varphi^{-1}(U_0)$, $\widetilde{V} = \psi^{-1}(V_0)$ and $\phi = F \circ \psi\colon \widetilde{V} \longrightarrow W$. Then $(\widetilde{V}, \phi)$ is a chart for N around $f(p)$ and

$$\phi(f(M) \cap \widetilde{V}) = F(\psi(f(\widetilde{U}))) = F(g(U_0)) = W \cap (\mathbb{R}^m \times \{0\}) = \phi(\widetilde{V}) \cap (\mathbb{R}^m \times \{0\})$$

as required.

Conversely, let $Y^m \subset N^n$ be a submanifold in the second sense. The inclusion $f\colon Y \hookrightarrow N$ is certainly a homeomorphism onto its image so we only need to check that its differential is injective at all points $p \in Y$. But choosing a chart (U, ϕ) for N around p with $\phi(p) = 0$ and

$$\phi(U \cap Y) = \phi(U) \cap (\mathbb{R}^m \times \{0\})$$

we obtain a commutative diagram

$$\begin{array}{ccc} T_p(U \cap Y) & \xrightarrow{\mathrm{d}f(p)} & T_pU \\ \mathrm{d}\phi(p)\downarrow & & \downarrow\mathrm{d}\phi(p) \\ T_0(\mathbb{R}^m \times 0) & \xrightarrow[\mathrm{d}g(0)]{} & T_0\mathbb{R}^n \end{array}$$

with $g\colon \mathbb{R}^m \times \{0\} \longrightarrow \mathbb{R}^n$ the inclusion map. Since $\mathrm{d}\phi(p)$ is an isomorphism and $\mathrm{d}g(0)$ is certainly injective we conclude that $\mathrm{d}f(p)$ must be injective as well.

Exercise 2.2. Prove the following result: if $f\colon M^m \longrightarrow N^n$ is a submersion between two smooth manifolds, or more generally if f is simply a smooth map and $y \in N$ is a regular value of f, then $S = f^{-1}(y)$ has the structure of a smooth submanifold of M of dimension $m - n$.

Solution. Take $p \in f^{-1}(y)$ and pick charts (U, ϕ) around p and (V, ψ) around y such that $\psi(y) = 0$. Write $g = \psi \circ f \circ \phi^{-1}$. Since p is a regular point of f, the image $\phi(p)$ is a regular point of g and by the implicit function theorem we can, by shrinking U if necessary, assume that there is a diffeomorphism $F\colon \phi(U) \longrightarrow W \subset \mathbb{R}^m$ such that

$$g(F^{-1}(x)) = (x_n, \ldots, x_m), \quad \text{for all } x = (x_1, \ldots, x_m) \in W.$$

Setting $\varphi = F \circ \phi$ we compute

$$\varphi(U \cap f^{-1}(y)) = F(\phi(U) \cap g^{-1}(0)) = \varphi(U) \cap F(g^{-1}(0)) = \varphi(U) \cap (\mathbb{R}^{m-n} \times \{0\}).$$

Since $p \in f^{-1}(y)$ was arbitrary, Exercise 2.1 implies that $f^{-1}(y)$ is a submanifold of M of dimension $m - n$.

Exercise 2.3. Prove that $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ can be given the structure of an n–dimensional manifold by exhibiting it as the regular value of some map.
Solution. Let $f \colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be given by $f(x) = \|x\|^2$. Then f is smooth and

$$\mathrm{d}f(x)(v) = 2\langle x, v\rangle \in \mathbb{R} = T_{f(x)}\mathbb{R}$$

for all $x \in \mathbb{R}^{n+1}$ and $v \in T_x\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$. In particular, if $x \in S^n = f^{-1}(1)$ we have $\mathrm{d}f(x)(x) = 2$ and therefore $\mathrm{d}f(x)$ is nonzero. Since $\mathbb{R}$ is 1–dimensional this means $\mathrm{d}f(x)$ is surjective, that is, x is a regular point of f. We conclude that 1 is a regular value of f and by Exercise 2.2 that S^n is a submanifold of $\mathbb{R}^{n+1}$ of dimension $n + 1 - 1 = n$.

Exercise 2.4. Let $M \subset \mathbb{R}^N$ be a submanifold. In class, we gave a first definition of the tangent space to M at a point p as follows: a vector $v \in \mathbb{R}^N$ is said to be tangent to M at p if there exists a smooth parametrized curve $\alpha \colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^N$ with image $\mathrm{im}(\alpha) \subset M$, $\alpha(0) = p$ and $\alpha'(0) = v$. The *tangent space* $T_pM \subset \mathbb{R}^N$ is then the set of all tangent vectors to M at p.

Prove that T_pM is a vector space (or equivalently, that $T_pM \subset \mathbb{R}^N$ is a linear subspace).
Solution. Take $v \in T_pM$ and choose a curve $\alpha \colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^N$ in M with $\alpha(0) = p$ and $\alpha'(0) = v$. Given any $\lambda \in \mathbb{R} \smallsetminus \{0\}$ the curve $\tilde{\alpha}(t) = \alpha(\lambda t)$ defined for $t \in (-\varepsilon/\lambda, \varepsilon/\lambda)$ satisfies $\tilde{\alpha}(0) = p$ and $\tilde{\alpha}'(0) = \lambda v$. So we only need to check that $v + w \in T_pM$ whenever $v, w \in T_pM$, that is, we need to find a curve $\gamma \colon (-\delta, \delta) \longrightarrow \mathbb{R}^N$ with image in M such that $\gamma(0) = p$ and $\gamma'(0) = v + w$.

Let (U, ϕ) be a chart for $\mathbb{R}^N$ such that $\phi(p) = 0$ and

$$\phi(U \cap M) = \phi(U) \cap (\mathbb{R}^n \times \{0\})$$

where $\dim(M) = n$. Let $\alpha \colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^N$ be a smooth curve in M with $\alpha(0) = p$. Then the image of $\phi \circ \alpha$ is contained i $\mathbb{R}^n \times \{0\}$ and consequently $(\varphi \circ \alpha)'(0) = \mathrm{d}\varphi(p)(\alpha'(0)) \in \mathbb{R}^n \times \{0\}$ as well. That is, $\mathrm{d}\varphi(p)(v), \mathrm{d}\varphi(p)(w) \in \mathbb{R}^n \times \{0\}$ for $v, w \in T_pM$. Since $\mathbb{R}^n \times \{0\}$ is a linear subspace of $\mathbb{R}^N$ we also have $\mathrm{d}\varphi(p)(v) + \mathrm{d}\varphi(p)(w) \in \mathbb{R}^n \times \{0\}$. Pick $\varepsilon > 0$ small enough so that $t(\mathrm{d}\varphi(p)(v) + \mathrm{d}\varphi(w)) \in \phi(U) \cap (\mathbb{R}^n \times \{0\})$ for $t \in (-\varepsilon, \varepsilon)$. Define the curve $\gamma \colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^N$ by setting

$$\gamma(t) = \varphi^{-1}(t(\mathrm{d}\varphi(p)(v) + \mathrm{d}\varphi(p)(w)))$$

and observer that $\mathrm{im}(\gamma) \subset M$ and $\gamma(0) = p$. Furthermore, by the chain rule

$$\gamma'(0) = \mathrm{d}(\varphi^{-1})(p)(\mathrm{d}\varphi(p)(v) + \mathrm{d}\varphi(p)(w)) = v + w$$

and we conclude $v + w \in T_pM$.

Exercise 2.5. Let $O(n) = \{A \in M_n(\mathbb{R}) : AA^T = I\}$ be the *orthogonal group*, where A^T is the *transpose* of A. Consider the map $\phi \colon M_n(\mathbb{R}) \longrightarrow \mathrm{Sym}(n)$ with $\phi(A) = AA^T$ where $\mathrm{Sym}(n) = \{B \in M_n(\mathbb{R}) : B = B^T\}$ is the set of *symmetric matrices*.

(i) Show that $\mathrm{Sym}(n)$ is a submanifold of $M_n(\mathbb{R})$ (and in particular a manifold), and compute its dimension.
(ii) Prove that $I \in \mathrm{Sym}(n)$ is a regular value of ϕ.
(iii) Prove that $O(n)$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension?
(iv) Prove that $O(n)$ is compact.

Solution. First, given real vector spaces $V \subset W$ we claim that W has a canonical smooth structure and with respect to this smooth structure V automatically becomes a smooth submanifold of W. Indeed, choose a linear isomorphism $\phi\colon W \longrightarrow \mathbb{R}^n$ and equip W with a topology so that ϕ becomes a homeomorphism. Then $\{(W, \phi)\}$ is a smooth atlas which determines a smooth structure on W. This smooth structure does not depend on the choice of ϕ: any other choice $\phi'\colon W \longrightarrow \mathbb{R}^n$ yields a compatible atlas $\{(W, \phi')\}$ because the transition maps $\phi \circ (\phi')^{-1}$ and $\phi' \circ \phi^{-1}$ are linear and therefore smooth. Given a linear subspace $V \subset W$ pick an isomorphism $V \longrightarrow \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$ and extend it to an isomorphism $\phi\colon W \longrightarrow \mathbb{R}^n$ so that the diagram

$$\begin{array}{ccc} V & \hookrightarrow & W \\ \downarrow & & \downarrow \phi \\ \mathbb{R}^m \times \{0\} & \hookrightarrow & \mathbb{R}^n \end{array}$$

commutes. Then (W, ϕ) is a chart for W which exhibits V as an m–dimensional submanifold by Exercise 2.1.

(i) Since $\mathrm{Sym}(n)$ is an $\binom{n+1}{2}$–dimensional linear subspace of the n^2–dimensional vector space $M_n(\mathbb{R})$ the preceding remark shows that it is an $\binom{n+1}{2}$–dimensional submanifold of $M_n(\mathbb{R})$.

(ii) Let $A, X \in M_n(\mathbb{R})$ be arbitrary. Then

$$\phi(A + X) = (A + X)(A + X)^T = AA^T + AX^T + XA^T + XX^T = AA^T + L_A(X) + o(\|X\|)$$

where $L_A\colon M_n(\mathbb{R}) \longrightarrow \mathrm{Sym}(n)$ is the linear map with $L_A(X) = AX^T + XA^T$. It follows that ϕ is differentiable at A with derivative L_A. Furthermore, we see that the assignment $A \longmapsto L_A$ is linear and therefore smooth. Hence ϕ is smooth itself.

We have verified that ϕ is in fact a smooth map with derivative $\mathrm{d}\phi(A) = L_A$. To see that I is a regular value of ϕ we need to check that $L_A\colon M_n(\mathbb{R}) \longrightarrow \mathrm{Sym}(n)$ is surjective whenever $AA^T = I$. But then, given any $E \in \mathrm{Sym}(n)$ we have

$$L_A(EA/2) = A(EA/2)^T + (EA/2)A^T = \frac{1}{2}\left(AA^T E^T + EAA^T\right) = E.$$

So L_A is indeed surjective.

(iii) Since $O(n) = \phi^{-1}(I)$ and we have seen that I is a regular value of ϕ, Exercise 2.2 implies that $O(n)$ is a submanifold of $M_n(\mathbb{R})$ with dimension

$$n^2 - \binom{n+1}{2} = \binom{n}{2}.$$

(iv) Since $O(n) = \phi^{-1}(I)$ is the preimage of the closed set $\{I\}$ under the continuous map ϕ, it is closed itself. Hence it will be enough to show that $O(n)$ is a bounded subset of $M_n(\mathbb{R})$. Let $\|\_\|$ be the operator norm on $M_n(\mathbb{R})$. For $A \in O(n)$ we then have $1 = \|I\| = \|AA^T\| = \|A\|^2$. Therefore $O(n)$ is a bounded set in the operator norm and because all norms on finite dimensional vector spaces induce the same topology this is enough to conclude that $O(n)$ is compact.

Exercise 2.6. Let Γ be a group and M a smooth manifold. A (C^∞) *action* of Γ on M is a group homomorphism ρ from Γ to the group $\mathrm{Diff}(M)$ of diffeomorphisms on M. If $\gamma \in \Gamma$ and $x \in M$ we write $\gamma x = \rho(\gamma)(x)$ for the image of x under the diffeomorphism $\rho(\gamma)$.

Recall from class that the *quotient space* M/Γ of the action Γ on M is the set of equivalence classes of the equivalence relation $\sim$ defined by $x \sim y$ iff $y = \gamma x$ for some $\gamma \in \Gamma$.

(i) We say the action of Γ on M is *discontinuous* if, for every compact subset K of M, the set $\{\gamma \in \Gamma : K \cap \gamma K \neq \emptyset\}$ is finite. We say the action of Γ on M is *free* if $\gamma x \neq x$ for every $x \in M$ and $\gamma \in \Gamma \setminus \{\mathrm{id}\}$. Prove that if Γ acts freely and discontinuously on M, then the quotient M/Γ naturally has the structure of smooth manifold.

(ii) Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ act on $S^n \subset \mathbb{R}^{n+1}$ by sending $x \longmapsto -x$. Using the standard manifold structure on S^n (either as given above via expressing S^n as a preimage or as studied on the homework last week), prove that $S^n/\mathbb{Z}_2$ has the structure of a manifold, which is diffeomorphic to $\mathbb{RP}^n$, equipped with the smooth manifold structure which you defined on your homework last week.

Solution. Since M is locally compact and Γ acts discontinuously on M, for every $x \in M$ there is a neighborhood U of x such that $\gamma U \cap U = \emptyset$ whenever $\gamma \neq \mathrm{id}$. It follows that the projection map $\pi\colon M \longrightarrow M/\Gamma$ restricts to a homeomorphism $\pi|_U\colon U \longrightarrow \pi(U)$. In particular, π is an open map and M/Γ is therefore second countable. Let $x, y \in M$ with $\pi(x) \neq \pi(y)$. Choose neighborhoods U_0 of x and V_0 of y such that $\overline{U_0}$ and $\overline{V_0}$ are compact. Then

$$\overline{U_0} \cap \gamma \overline{V_0} \subset (\overline{U_0} \cup \overline{V_0}) \cap \gamma(\overline{U_0} \cup \overline{V_0}) \neq \emptyset$$

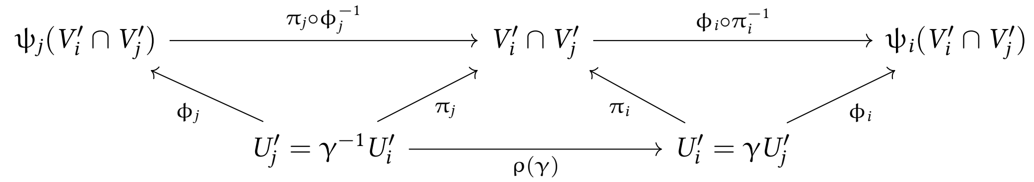
for only finitely many $\gamma \in \Gamma$. Therefore $U = U_0 \smallsetminus \bigcup_\gamma \gamma \overline{V_0}$ is open and $x \in U$ since $\pi(x) \neq \pi(y)$. Similarly, $V = V_0 \smallsetminus \bigcup_\gamma \gamma \overline{U_0}$ is an open neighborhood of y. Furthermore, we have arranged that $\pi(U) \cap \pi(V) = \emptyset$ and so we conclude that M/Γ is Hausdorff. It remains to construct a smooth atlas on M/Γ.

Let $\{(U_i, \phi_i)\}_i$ be a smooth atlas for M and assume without loss of generality that the U_i are small enough such that $\pi_i = \pi|_{U_i}\colon U_i \longrightarrow \pi(U_i)$ is a homeomorphism. We claim then that $\{(\pi(U_i), \phi_i \circ \pi_i^{-1})\}$ is a smooth atlas for M/Γ. The maps $\psi_i = \phi_i \circ \pi_i^{-1}$ certainly are homeomorphism from $V_i = \pi(U_i)$ to some open set in $\mathbb{R}^n$ so we only need to check that the transition maps are smooth. That is, for indices i and j we need to show that the map

$$\psi_i \circ \psi_j^{-1}\colon \psi_j(V_i \cap V_j) \longrightarrow \psi_i(V_i \cap V_j)$$

is smooth. Let $x \in V_i \cap V_j$. Then there are unique $u_i \in U_i$ and $u_j \in U_j$ such that $\pi(u_i) = x = \pi(u_j)$. In general, u_i and u_j need not be equal! However, there always is some $\gamma \in \Gamma$ such that $u_i = \gamma u_j$. Let $U_i' = U_i \cap \gamma U_j$ and $U_j' = \gamma^{-1} U_i \cap U_j$ and denote their images under π by $V_i' = \pi(U_i')$ and $V_j' = \pi(U_j')$.

Let $p \in U_i' = U_i \cap \gamma U_j$, say $p = \gamma q$ with $q \in U_j$. Then $\pi(p) = \pi(q)$ and therefore $\pi_i^{-1}(\pi_j(p)) = q = \gamma p$. This shows that $\pi_j^{-1} \circ \pi_i = \rho(\gamma)\colon U_i' \longrightarrow U_j' = \gamma U_i'$. We can conclude that the diagram



commutes. This says that, on a neighborhood of $\psi_j(x)$, the transition map $\psi_i \circ \psi_j^{-1}$ coincides with the map $\phi_i \circ \rho(\gamma) \circ \phi_j^{-1}$ which is smooth sind Γ acts via diffeomorphisms. In particular, $\psi_i \circ \psi_j^{-1}$ is smooth at $\psi_j(x)$. Since $x \in V_i \cap V_j$ was arbitrary we conclude that $\{(\pi(U_i), \psi_i)\}$ is a smooth atlas on M/Γ.

Note that the construction of the smooth structure on M/Γ immediately implies the following universal property: Given any smooth function $f\colon M \longrightarrow N$ which is Γ–invariant in the sense that $f \circ \rho(\gamma) = f$ for all $\gamma \in \Gamma$, there is a unique smooth function $g\colon M/\Gamma \longrightarrow N$ such that $g \circ \pi = f$.

Concerning $\mathbb{RP}^n$, because $\mathbb{Z}_2$ is finite it automatically acts discontinuously on S^n. Furthermore, the action is certainly free and so we conclude that $S^n/\mathbb{Z}_2$ is a smooth manifold in a natural way. By the definition of $\mathbb{RP}^n$ we have a smooth projection $g\colon \mathbb{R}^{n+1} \smallsetminus \{0\} \longrightarrow \mathbb{RP}^n$. Restricting this to the submanifold $S^n \subset \mathbb{R}^{n+1}$ we obtain a smooth map $f\colon S^n \longrightarrow \mathbb{RP}^n$. Let's compute the differential of g at the north pole $N = (1, 0, \dots, 0) \in S^n$. A convenient chart around $g(N)$ is given by

$$\phi_0\colon U_0 = \{[x_0 : \cdots : x_n] : x_0 = 1\} \longrightarrow \mathbb{R}^n, [x_0 : \cdots : x_n] \longmapsto (x_1, \dots, x_n)$$

and the coordinate expression for g with respect to this chart is simply

$$\phi_0 \circ g\colon g^{-1}(U_0) \longrightarrow \mathbb{R}^n, (x_0, \dots, x_n) \longmapsto (x_1, \dots, x_n).$$

We see that in these coordinates the derivative of f, the restriction of g to S^n, is simply the identity map. In particular, this implies that f is a local diffeomorphism near N.

We have an evident smooth action of the orthogonal group $O(n+1)$ on S^n and on $\mathbb{RP}^n$ and this action is easily seen to be transitive. Furthermore, f is equivariant with respect to these actions. This implies that f is in fact a local diffeomorphism everywhere on S^n, not just at the north pole.

The last step is to descend f to a smooth map $\overline{f}\colon S^n/\mathbb{Z}_2 \longrightarrow \mathbb{RP}^n$. By the universal property of the quotient $S^n/\mathbb{Z}_2$ this is possibly if and only if $f(x) = f(-x)$. But this is immediate from the definition of f. Additionally, if $\overline{f}([x]) = \overline{f}([y])$ for two points $x, y \in S^n$ then we must have $x = \lambda y$ for some scalar $\lambda \in \mathbb{R} \setminus \{0\}$. But then the only possibilities for λ are ± 1 and consequently $[x] = [y]$. Therefore $\overline{f}$ is actually bijective and we saw already that it is a local diffeomorphism. But a bijective local diffeomorphism is a diffeomorphism. Hence, $S^n/\mathbb{Z}_2 \cong \mathbb{RP}^n$.