Math 535a Homework 3

Due Monday, February 13, 2017 by 5 pm

Please remember to write down your name on your assignment.

1. Give a detailed proof of the equivalence between the three definitions of T_pM given in class. Then, prove that the construction of the derivative
$$df_p : T_pM \to T_{f(p)}N$$
is the same for the three definitions, meaning the following: If $T_p^{(i)}M$ denotes the ith construction of the tangent space, for $i = 1, 2, 3$, and
$$df_p^{(i)} : T_p^{(i)}M \to T_{f(p)}^{(i)}N$$
the corresponding three different constructions of the derivative, then show that for any M and p and any i, j there are isomorphisms
$$g_{p,M}^{(ij)} : T_p^{(i)}M \cong T_p^{(j)}M$$
which intertwine the derivative maps, in the sense that $df_p^{(i)} = g_{f(p),N}^{(ji)} \circ df_p^{(j)} \circ g_{p,M}^{(ij)}$ (where $g_{p,M}^{(ji)} = (g_{p,M}^{(ij)})^{-1}$).

2. Let $M = f^{-1}(y)$ be the preimage of a regular value $y \in \mathbb{R}^{N-m}$ of a smooth function $f : \mathbb{R}^N \to \mathbb{R}^{N-m}$. (for instance, $M = S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3 = f^{-1}(1)$, where $f : (x, y, z) \mapsto x^2 + y^2 + z^2$).

   (a) Let $\widetilde{TM} = \{(x, v) \in \mathbb{R}^N \times \mathbb{R}^N | x \in M, v \in \ker df_x\}$. Show that as defined, $\widetilde{TM}$ is a smooth submanifold of $\mathbb{R}^N \times \mathbb{R}^N$ of dimension $2m$ (where M is an m-dimensional manifold).

   (b) Prove that there is a diffeomorphism between $\widetilde{TM}$ and the *tangent bundle of M* as defined in class:
   $$\widetilde{TM} \cong TM$$
   (It follows that, for instance, $TS^2 \cong \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 | x \in S^2 | v \cdot x = 0\}$.

3. Let M^m be a manifold of dimension m and $p \in M$ a point. Recall that $\mathcal{F}_p \subset C^\infty(p)$ is the ideal of germs of functions on M which vanish at $p \in M$. Let $\mathcal{F}_p^k$ be the ideal of $C^\infty(p)$ generated by $f_1 \cdots f_k$, where $f_i \in \mathcal{F}_p$. (This means that every element of $\mathcal{F}_p^k$ is a sum $\sum_i g_i f_{1i} \cdots f_{ki}$, where $g^i \in C^\infty(p)$, and $f_{ij} \in \mathcal{F}_p$).

   (a) Prove that, in every set of local coordinates $(x_1, \ldots, x_k)$ around the point p, an element $f \in \mathcal{F}_p^k$ has a Taylor expansion which vanishes to order k. You may assume a version of Taylor's approximation theorem stated in class.

   (b) Compute the dimension of $\mathcal{F}_p^k / \mathcal{F}_p^{k+1}$.

(c) Construct a smooth manifold along with a map to M, $E \xrightarrow{\pi} M$ whose "fiber" $E_p = \pi^{-1}(p)$ at the point $p \in M$ is $\mathcal{F}^1_p/\mathcal{F}^3_p$.

4. Let $f : M \to N$ be a smooth map between manifolds. Prove that the following diagram commutes:

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{f^*} & \Omega^0(M) \\ \downarrow d & & \downarrow d \\ \Omega^1(N) & \xrightarrow{f^*} & \Omega^1(M) \end{array}$$

5. Give a detailed proof that the cotangent bundle T^*M is a smooth manifold and that the projection map $\pi : T^*M \to M$ is a smooth map.

6. Let f and g be smooth real-valued functions on a manifold M. Prove that $d(fg) = fdg + gdf$.

7. Let $i : S^1 = [0, 2\pi]/(0 \sim 2\pi) \to \mathbb{R}^2$ be the map $\theta \mapsto (\cos(\theta), \sin(\theta))$. Compute $i^*((x^2 + y)dx + (3 + xy^2)dy)$.[1]

8. Earlier in class, we defined the notion of a *category* $\mathcal{C}$; examples given include *topological spaces* **Top**, and *vector spaces* **Vect**.

   A *functor* $F : \mathcal{C} \to \mathcal{D}$ from category $\mathcal{C}$ to $\mathcal{D}$ is an assignment, to every object of $\mathcal{C}$, an object of $\mathcal{D}$, and an induced map on morphism spaces. More precisely, a *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is specified by the following data:
   - A map on object $F : \text{ob } \mathcal{C} \to \text{ob } \mathcal{D}$
   - For every pair of objects X, Y, a map on morphism spaces $F = F_{XY} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$, which satisfies:
     - F sends identity morphisms to identity morphisms (so $F(id_X) = id_{F(X)}$, where $X \in \text{ob } \mathcal{C}$.), and
     - F is compatible with compositions, in the sense that $F(g) \circ F(f) = F(g \circ f)$ for any objects X, Y, Z and morphisms $g \in \hom(Y, Z)$, $f \in \hom(X, Y)$.

   A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$, written as

$$G : \mathcal{C}^{op} \to \mathcal{D},$$

   consists of the following data: [2]
   - A map on object $G : \text{ob } \mathcal{C} \to \text{ob } \mathcal{D}$
   - For every pair of objects X, Y, a map on morphism spaces $G = G_{XY} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(G(Y), G(X))$ (note the order reversal), which satisfies:
     - G sends identity morphisms to identity morphisms (so $G(id_X) = id_{G(X)}$, where $X \in \text{ob } \mathcal{C}$.), and

[1] As discussed in class, the notation $f_1dx + f_2dy$, where f_1 and f_2 are smooth functions on $\mathbb{R}^2$, is a common shorthand for the 1-form $\mathbb{R}^2 \to T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ sending $\vec{x}$ to $(\vec{x}, (f_1(\vec{x})dx + f_2(\vec{x})dy))$.

[2] A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is the same as a covariant functor from the *opposite category* $\mathcal{C}^{op}$ of $\mathcal{C}$ to $\mathcal{D}$, hence the notation. We will not elaborate on this point more here.

– G is compatible with compositions, in the sense that $G(f) \circ G(g) = G(g \circ f)$ for any objects X, Y, Z and morphisms $g \in \hom(Y, Z)$, $f \in \hom(X, Y)$.

In other words, a contravariant functor is specified by the same sort of data as a covariant functor, except the order of morphisms in the target is reversed in passing from the source to the target category.

(a) To any topological space M, define a category **Open**(M) as follows:
- objects of **Open**(M) are the open subsets $U \subset M$.
- Morphisms from U to V are *inclusions*, meaning that: if U is not contained in V, then $\hom(U, V) = \emptyset$, and if $U \subset V$, then $\hom(U, V) = \{i_{UV} : U \hookrightarrow V\}$, where i_{UV} simply denotes the inclusion map $U \hookrightarrow V$.
- Composition of morphisms $\hom(V, W) \times \hom(U, V) \to \hom(U, W)$ (which is only non-trivial if $U \subset V \subset W$) is the usual composition of inclusions. Namely $i_{VW} \circ i_{UV} = i_{UW}$.

Verify that **Open**(M) satisfies the axioms of a category.

(b) A **pre-sheaf** on M taking values in a category $\mathcal{C}$ is a functor

$$F : \mathbf{Open}(M)^{op} \to \mathcal{C}.$$

For instance, if $\mathbf{Alg}_{\mathbb{R}}$ denotes the category of $\mathbb{R}$-algebras (objects are $\mathbb{R}$ algebras,[3] and morphisms are $\mathbb{R}$-algebra homomorphisms[4], then a *pre-sheaf of* $\mathbb{R}$*-algebras* on M is a functor $F : \mathbf{Open}(M) \to \mathbf{Alg}_{\mathbb{R}}$.

Let M be a smooth manifold now, and define a functor $C^\infty(-) : \mathbf{Open}(M)^{op} \to \mathbf{Alg}_{\mathbb{R}}$ by, on objects

$$U \to C^\infty(U),$$

and on the inclusions $i_{UV} : U \to V$, the induced map $C^\infty(-)_{UV}(i_{UV}) \in \hom_{\mathbf{Alg}_{\mathbb{R}}}(C^\infty(V), C^\infty(U))$ is the restriction map on functions. $i_{UV}^* : C^\infty(V) \to C^\infty(U)$.

Verify that $C^\infty(-)$ is indeed a pre-sheaf of algebras, and in particular a contravariant functor.

(c) Verify that the notion of a pre-sheaf of algebras $\mathcal{F}$ is equivalent to the following data:
- For every open set $U \in M$, an algebra $\mathcal{F}(U)$.
- For every inclusion of open sets $U \subseteq V$, a restriction map $\rho_{U \subset V} : \mathcal{F}(V) \to \mathcal{F}(U)$, satisfying, $\rho_{U \subset U} = id_{\mathcal{F}(U)}$, and for any triple $U \subset V \subset W$, that $\rho_{U \subset V} \circ \rho_{V \subset W} =$

[3]Let k be any field. For our purposes, a *k-algebra* A is a vector space over k equipped with a multiplication map $\times : A \times A \to A$ which is a bilinear map. We further assume that the mutiplication map is associative, and that there is a multiplicative identity $1 \in A$ satisfying $1 \cdot \alpha = \alpha \cdot 1 = \alpha$, for $\alpha \in A$ (elsewhere, such A are frequently called *associative unital algebras*). You should verify for yourself that if U is any manifold, then $C^\infty(U)$ is an $\mathbb{R}$-algebra in this sense.

[4]For our purposes, an *k-algebra homomorphism* $F : A \to B$ is a linear map of vector spaces which is compatible with the multiplication maps, meaning that $F(\alpha \cdot \beta) = F(\alpha) \cdot F(\beta)$. F should also preserve the identity elements, so $F(1) = 1$; this is frequently elsewhere called a *unital algebra homomorphism.* You should verify for yourself that if $f : M \to N$ is any C^∞ map, then the pullback $f^* : C^\infty(N) \to C^\infty(M)$ is an $\mathbb{R}$-algebra homomorphism

$\rho_{U \subset W}$.

(d) A pre-sheaf as defined in the previous section is said to be a *sheaf* if for any pair of open sets U, V, whenever there is an element $f_1 \in \mathcal{F}(U)$ and an element $f_2 \in \mathcal{F}(V)$ with the same restriction on the overlapping region,[5] then there *exists* a *unique* element $g \in \mathcal{F}(U \cup V)$ restricting to f_1 and f_2 on U and V.[6]

Let M be a manifold. Verify that the pre-sheaf on M, $C^\infty(-)$ defined above is in fact a sheaf.

[5] meaning that $\rho_{U \cap V \subset U}(f_1) = \rho_{U \cap V \subset V}(f_2)$

[6] meaning that $\rho_{U \subset U \cup V}(g) = f_1$, $\rho_{U \subset U \cup V}(g) = f_2$.