## Homework 3

Exercise 3.1. Give a detailed proof of the equivalence between the three definitions of $T_pM$ given in class. Then, prove that the construction of the derivative $\mathrm{d}f_p\colon T_pM \longrightarrow T_{f(p)}N$ is the same for the three definitions, meaning the following: If $T_p^{(i)}M$ denotes the three different constructions of the tangent space, for $i = 1, 2, 3$, and $\mathrm{d}f_p^{(i)}$ the corresponding three different constructions of the derivative, then show that the isomorphisms $g_{(ij)}\colon T_p^{(i)}M \xrightarrow{\sim} T_p^{(j)}M$ intertwine the derivative maps, in the sense that $\mathrm{d}f_p^{(i)} = g_{(ji)} \circ \mathrm{d}f_p^{(j)} \circ g_{(ij)}$ and $g_{(ji)} = g_{(ij)}^{-1}$.

*Solution.* Fix a coordinate chart $(U, (x^1, \dots, x^n))$ centered at $p$. We first remark that given any derivation $D \in \mathrm{Der}(C^\infty(p), \mathbb{R})$ and a smooth function $f\colon U \longrightarrow \mathbb{R}$ we can expand $f$ around $p$ using Taylor's theorem:

$$f(q) = f(p) + \sum_i a_i x^i(q) + \sum_{ij} a_{ij}(q) x^i(q) x^j(q) \qquad (\star)$$

for $q \in U$ and smooth functions $a_{ij}\colon U \longrightarrow \mathbb{R}$. Applying $D$ we find

$$D(f) = \sum_i a_i D(x^i) + \sum_{ij} D(a_{ij}) x^i(p) x^j(p) + \sum_{ij} a_{ij}(p) D(x^i) x^j(p) + \sum_{ij} a_{ij}(p) x^i(p) D(x^j) = \sum_i a_i D(x^i) \quad (\star\star)$$

because $x^i(p) = 0$.

To fix notation let $T_p^{(1)}M = C_p/\sim$, $T_p^{(2)}M = \mathrm{Der}(C^\infty(p), \mathbb{R})$ and $T_p^{(3)}M = (\mathscr{F}_p/\mathscr{F}_p^2)^\vee$. Given a small curve $\alpha \in C_p$ through $p$ define a derivation $D_\alpha\colon C^\infty(p) \longrightarrow \mathbb{R}$ by $D_\alpha([f]) = (f \circ \alpha)'(0)$. This is well-defined since the expression $(f \circ \alpha)'(0)$ depends only on the germ of $f$ at $\alpha(0) = p$. Similarly, $D_\alpha$ only depends on the equivalence class of $\alpha$. Also, $D_\alpha$ is evidently $\mathbb{R}$-linear and satisfies the Leibniz rule. So $D_\alpha \in \mathrm{Der}(C^\infty(p), \mathbb{R})$ and we define $g_{(12)}([\alpha]) = D_\alpha$ to obtain a linear map $g_{(12)}\colon T_p^{(1)}M \longrightarrow T_p^{(2)}M$.

To define $g_{(21)}\colon T_p^{(2)}M \longrightarrow T_p^{(1)}M$ let $D \in \mathrm{Der}(C^\infty(p), \mathbb{R})$. Define a curve $\alpha_D\colon I \longrightarrow M$ with coordinates $x^i(\alpha_D(t)) = D(x^i)t$. Then $\alpha_D \in C_p$ and its equivalence class $[\alpha_D]$ depends linearly on $D$. Set $g_{(21)}(D) = [\alpha_D]$.

To define $g_{(23)}$, let $D \in \mathrm{Der}(C^\infty(p), \mathbb{R})$ and $[f] \in \mathscr{F}_p/\mathscr{F}_p^2$. Note that $D(fg) = f(p)D(g) + D(f)g(p) = 0$ for $f, g \in \mathscr{F}_p$. Hence, $D$ defines a map $D\colon \mathscr{F}_p/\mathscr{F}_p^2 \longrightarrow \mathbb{R}$ which we take as $g_{(23)}(D)$.

To define $g_{(32)}$, let $G\colon \mathscr{F}_p/\mathscr{F}_p^2 \longrightarrow \mathbb{R}$ be linear and $f \in C^\infty(p)$. Then $f - f(p)$ vanishes at $p$ and so we can consider $[f - f(p)] \in \mathscr{F}_p/\mathscr{F}_p^2$. Set $D_G(f) = G([f - f(p)])$. Then $D_G$ is clearly $\mathbb{R}$–linear and

$$\begin{aligned} D_G(fg) = D_G(fg) &= G([fg - f(p)g(p)]) = G([fg - (f - f(p))(g - g(p)) - f(p)g(p)]) = \\ &= G([f(p)g - f(p)g(p) + g(p)f - g(p)f(p)]) = f(p)D_G(g) + g(p)D_G(f). \end{aligned}$$

Hence, $D_G \in \mathrm{Der}(C^\infty(p), \mathbb{R})$ and $D_G$ depends linearly on $G$. So we can take $g_{(32)}(G) = D_G$.

Finally, we are forced to define $g_{(13)} = g_{(23)} \circ g_{(12)}$ and $g_{(31)} = g_{(21)} \circ g_{(32)}$. To check that $g_{(21)} \circ g_{(12)} = \mathrm{id}$ let $\alpha\colon I \longrightarrow M$ be some curve with $\alpha(0) = p$ and let $f\colon M \longrightarrow \mathbb{R}$ be some smooth function. Expand $f$ as in ($\star$). We compute

$$(f \circ \alpha)'(0) = \sum_i a_i (x^i \circ \alpha)'(0)$$

and

$$(f \circ \alpha_{D_\alpha})'(0) = \sum_i a_i (x^i \circ \alpha_{D_\alpha})'(0) = \sum_i a_i D_\alpha(x^i) = \sum_i a_i (x^i \circ \alpha)'(0).$$

Hence $[a] = [\alpha_{D_\alpha}] = g_{(21)}(g_{(12)}([\alpha])) \in C_p/\sim$. Conversely, let $D \in \mathrm{Der}(C^\infty(p), \mathbb{R})$. Let $f \in C^\infty(p)$ and expand $f$ as in ($\star$). Then

$$D_{\alpha_D}(f) = (f \circ \alpha_D)'(0) = \sum_i a_i (x^i \circ \alpha_D)'(0) = \sum_i a_i D(x^i) \overset{(\star\star)}{=} D(f).$$

We conclude that indeed $g_{(12)}^{-1} = g_{(21)}$.

To show that $g_{(23)} \circ g_{(32)} = \mathrm{id}$ let $D \in \mathrm{Der}(C^\infty(p), \mathbb{R})$ and $f \in C^\infty(p)$. We compute

$$g_{(23)}(g_{(32)}(D))(f) = D_D(f) = D([f - f(p)]) = D(f - f(p)) = D(f)$$

and conclude $D_D = D$. Conversely, let $G \in (\mathscr{F}_p/\mathscr{F}_p^2)^\vee$ and consider $D_G\colon \mathscr{F}_p/\mathscr{F}_p^2 \longrightarrow \mathbb{R}$. Take $[f] \in \mathscr{F}_p/\mathscr{F}_p^2$. Then

$$D_G([f]) = G([f - f(p)]) = G([f])$$

by definiton and we conclude that $g_{(23)}^{-1} = g_{(32)}$.

Lastly, we prove that the $g_{(ij)}$ intertwine the different definitions of the derivative $\mathrm{d}f_p^{(i)}$. Let $f\colon M \longrightarrow N$ be a smooth map. We first show that $\mathrm{d}f_p^{(2)} = g_{(12)} \circ \mathrm{d}f_p^{(1)} \circ g_{(21)}$. For this, let $D \in \mathrm{Der}(C^\infty(p), \mathbb{R})$ be a derivation and $g\colon N \longrightarrow \mathbb{R}$ smooth map. Then we have

$$\mathrm{d}f_p^{(2)}(D)(g) = D(g \circ f)$$

and

$$(g_{(12)} \circ \mathrm{d}f_p^{(1)} \circ g_{(21)})(D)(g) = (g \circ (f \circ \alpha_D))'(0) = ((g \circ f) \circ \alpha_D)'(0) = D(g \circ f).$$

To see that $\mathrm{d}f_p^{(3)} = g_{(23)} \circ \mathrm{d}f_p^{(2)} \circ g_{(32)}$ let $G \in (\mathscr{F}_p/\mathscr{F}_p^2)^\vee$ and $[g] \in \mathscr{F}_{N,f(p)}/\mathscr{F}_{N,f(p)}^2$. Then

$$\mathrm{d}f_p^{(3)}(G)([g]) = G([g \circ f])$$

and

$$(g_{(23)} \circ \mathrm{d}f_p^{(2)} \circ g_{(32)})(G)([g]) = D_G([g \circ f]) = G([g \circ f - g(f(p))]) = G([g \circ f]).$$

The other cases follow immediately from these by pre- and post-composing with $g_{(ij)}$.

Exercise 3.2. Let $M = f^{-1}(y)$ for a regular value $y \in \mathbb{R}^{N-m}$ of a smooth function $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^{N-m}$; for instance, $M = S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ is $f^{-1}(1)$ for $f(x, y, z) = x^2 + y^2 + z^2$.

(i) Let $\widetilde{TM} = \{(x, v) \in \mathbb{R}^N \times \mathbb{R}^N : x \in M,\ v \in \ker \mathrm{d}f_x\}$. Show that, as defined, $\widetilde{TM}$ is a smooth submanifold of $\mathbb{R}^N \times \mathbb{R}^N$ of dimension $2m$, where $\dim(M) = m$.

(ii) Prove that there is a diffeomorphism between $\widetilde{TM}$ and the *tangent bundle* of $M$ as defined in class: $\widetilde{TM} \cong TM$. It follows that, for instance, $TS^2 \cong \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \in S^2,\ \langle v, x\rangle = 0\}$.

*Solution.* First, consider the map $F\colon \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^{N-m} \times \mathbb{R}^{N-m}$ defined by $F(x, v) = (f(x), \mathrm{d}f_x(v))$. Then the fiber $F^{-1}(y, 0)$ is precisely $\widetilde{TM}$ and we only need to show that $\mathrm{d}F_{(x,v)}$ is surjective for all $(x, v) \in \widetilde{TM}$. We have

$$\mathrm{d}F_{(x,v)}(\xi, \zeta) = (\mathrm{d}f_x(\xi), \mathrm{d}^2 f_x(v, \xi) + \mathrm{d}f_x(\zeta))$$

where the notation $\mathrm{d}^2 f_x(v, \xi)$ means the vector in $\mathbb{R}^{N-m}$ with components

$$\left(\mathrm{d}^2 f_x(v, \xi)\right)^i = \sum_{j,k} \frac{\partial^2 f^i(x)}{\partial x^j \partial x^k} v^k \xi^j.$$

Let $(a, b) \in \mathbb{R}^{N-m} \times \mathbb{R}^{N-m}$. Since $(x, v) \in \widetilde{TM}$ the point $x \in M$ is a regular point for $f$ and there is some $\xi \in \mathbb{R}^N$ with $\mathrm{d}f_x(\xi) = a$. Similarly, there is some $\zeta \in \mathbb{R}^N$, depending on $\xi$, with $\mathrm{d}f_x(\zeta) = b - \mathrm{d}^2 f_x(v, \xi)$. Then $\mathrm{d}F_{(x,v)}(\xi, \zeta) = (a, b)$ and we conclude that $(x, v)$ is a regular point of $F$. Hence, $(y, 0)$ is a regular value of $F$ and the fiber $\widetilde{TM} = F^{-1}(y, 0)$ is a submanifold of $\mathbb{R}^N \times \mathbb{R}^N$ of dimension $2N - 2(N - m) = 2m$.

Let $\iota\colon M \longrightarrow \mathbb{R}^N$ denote the inclusion map. Its differential $\mathrm{d}\iota$ is a smooth embedding $TM \hookrightarrow \mathbb{R}^N \times \mathbb{R}^N$. So we only need to show that the image of $\mathrm{d}\iota$ is precisely $\widetilde{TM}$, that is, we need to show that $\mathrm{d}f_x(v) = 0$

for $x \in M$ and $v \in \mathbb{R}^N$ if and only if $v \in \operatorname{im}(d\iota_x)$. Assume first that $w \in T_xM$ is represented by a smooth curve $\alpha\colon (-\varepsilon, \varepsilon) \longrightarrow M$ with $\alpha(0) = x$. Then $d\iota_x(w) \in T_x\mathbb{R}^N = \mathbb{R}^N$ is represented by $\alpha$ considered as a map $(-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^N$ and $df_x(d\iota_x(w))$ is represented by the curve $f \circ \alpha\colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^{N-m}$. But since $\alpha$ lies entirely in $M$ the curve $f \circ \alpha$ is just the constant curve at $y$. That is, $df_x(d\iota_x(w)) = 0$ and we conclude $df_x(\operatorname{im}(d\iota_x)) = 0$,i. e. $\operatorname{im}(d\iota_x) \subset \ker(df_x)$. On the other hand, $\dim(\operatorname{im} d\iota_x) = m$ since $d\iota_x$ is injective. Since $df_x$ is surjective we also have $\dim(\ker(df_x)) = m$, so $\operatorname{im}(d\iota_x) = \ker(df_x)$ and we are done.

Exercise 3.3. Let $M^m$ be a manifold of dimension $m$ and $p \in M$ a point. Recall that $\mathscr{F}_p \subset C^\infty(p)$ is the ideal of germs of functions on $M$ which vanish at $p \in M$. Let $\mathscr{F}_p^k$ be the ideal of $C^\infty(p)$ generated by products $f_1 \cdots f_k$ for $f_i \in \mathscr{F}_p$. This means that every element of $\mathscr{F}_p^k$ is a sum $\sum_i g_i f_{1i} \cdots f_{ki}$ with $g_i \in C^\infty(p)$ and $f_{ij} \in \mathscr{F}_p$.

(i) Prove that, in every set of local coordinates $(x_1, \ldots, x_m)$ around the point $p$, an element $f \in \mathscr{F}_p^k$ has a Taylor expansion which vanishes to order $k$.
(ii) Compute the dimension of $\mathscr{F}_p^k / \mathscr{F}_p^{k+1}$.
(iii) Construct a smooth manifold along with a map to $M$, say $\pi\colon E \longrightarrow M$ whose fiber $E_p \in \pi^{-1}(p)$ at any point $p \in M$ is $\mathscr{F}_p / \mathscr{F}_p^3$.

*Solution.* Let $f^1, \ldots, f^k \in \mathscr{F}_p$, $g \in C^\infty(p)$ and let $(U, (x^1, \ldots, x^m))$ be a coordinate chart centered at $p$. We can expand each $f^i$ and $g$ by Taylor's theorem:

$$f^i(q) = \sum_j a_j^i x^j(q) + \sum_{jk} a_{jk}^i(q) x^j(q) x^k(q)$$

and

$$g(q) = g(p) + \sum_j a_j^0 x^j(q) + \sum_{jk} a_{jk}^0 x^j(q) x^k(q)$$

for $q \in U$ and smooth functions $a_{jk}^i\colon U \longrightarrow \mathbb{R}$. We compute for the product $gf^1 \cdots f^k$ that

$$\begin{aligned}(gf^1 \cdots f^k)(q) &= g(q) \prod_i \left( \sum_j a_j^i x^j(q) + \sum_{jk} a_{jk}^i(q) x^j(q) x^k(q) \right) = \\ &= \sum_{j_1, \ldots, j_k} g(p) a_{j_1}^1 \cdots a_{j_k}^k x^{j_1}(q) \cdots x^{j_k}(q) + \sum_{|\mathbf{i}| = k+1} x^{\mathbf{i}}(q) b_{\mathbf{i}}(q)\end{aligned}$$

for some smooth functions $b_{\mathbf{i}}\colon U \longrightarrow \mathbb{R}$.

From this expansion of $gf^1 \cdots f^k$ we can see that

$$gf^1 \cdots f^k \equiv \sum_{j_1, \ldots, j_k} g(p) a_{j_1}^1 \cdots a_{j_k}^k x^{j_1} \cdots x^{j_k} \pmod{\mathscr{F}_p^{k+1}}$$

We conclude that $\mathscr{F}_p^k / \mathscr{F}_p^{k+1}$ is generated by $\{x^{j_1} \cdots x^{j_k} : j_1 \leq j_2 \leq \cdots \leq j_k\}$. This set is linearly independent in $\mathscr{F}_p^k / \mathscr{F}_p^{k+1}$ and therefore the dimension of this space is $\binom{m+k-1}{k}$.

Lastly, note that $\mathscr{F}_p / \mathscr{F}_p^3 \cong \mathscr{F}_p / \mathscr{F}_p^2 \oplus \mathscr{F}_p^2 / \mathscr{F}_p^3$; we will suppress this isomorphism from the notation. Choose a countable atlas $\{(U_i, \phi_i = (x_i^1, \ldots, x_i^m))\}_{i \in I}$ for $M$ and define

$$E = \bigsqcup_{p \in M} \mathscr{F}_p / \mathscr{F}_p^2 \oplus \mathscr{F}_p^2 / \mathscr{F}_p^3.$$

There is an evident map $\pi\colon E \longrightarrow M$ and $E$ is covered by $\{\pi^{-1}(U_i)\}_{i\in I}$. Define maps

$$\psi_i\colon \pi^{-1}(U_i) \longrightarrow \phi_i(U_i) \times \mathbb{R}^m \times \mathbb{R}^{\binom{m+1}{2}}$$

as follows. Given $(p, f^1, f^2) \in \pi^{-1}(U_i)$ write $f^1$ and $f^2$ in our coordinates from (ii) as

$$f^1 = \sum_j a_j(x_i^j - p_i^j) \quad \text{and} \quad f^2 = \sum_{j\le k} b_{jk}(x_i^j - p_i^j)(x_i^k - p_i^k)$$

where $p_i^j = x_i^j(p)$ and set $\psi_i(p, f^1, f^2) = (\phi_i(p), (a_j)_j, (b_{jk})_{j\le k})$. There is a unique topology on $E$ making the $\psi_i$ into homeomorphisms, so we obtain an atlas $\{(\pi^{-1}(U_i), \psi_i)\}$ for $E$. To check that this atlas is smooth we compute the transition maps. Let $p \in U_i \cap U_j$ and $(f^1, f^2) \in \mathscr{F}_p/\mathscr{F}_p^2 \oplus \mathscr{F}_p^2/\mathscr{F}_p^3$. Write

$$f_1 = \sum_k a_k^i(x_i^k - p_i^k) \qquad f_2 = \sum_{k\le\ell} b_{k\ell}^i(x_i^k - p_i^k)(x_i^\ell - p_i^\ell)$$

and

$$f_1 = \sum_k a_k^j(x_j^k - p_j^k) \qquad f_2 = \sum_{k\le\ell} b_{k\ell}^j(x_j^k - p_j^k)(x_j^\ell - p_j^\ell).$$

Then the transition map $\psi_j \circ \psi_i^{-1}$ is given by

$$\psi_j(\psi_i^{-1}(\phi_i(p), (a_j^i)_j, (b_{k\ell}^i)_{k\le\ell})) = (\phi_j(p), (a_k^j)_k, (b_{k\ell}^j)_{k\le\ell}).$$

But

$$a_k^j = \left.\frac{\partial}{\partial x_j^k}\right|_p f_1 = \sum_\alpha a_\alpha^i \frac{\partial x_i^\alpha}{\partial x_j^k}(p)$$

and

$$b_{k\ell}^j = \frac{1}{2}\left.\frac{\partial^2}{\partial x_j^k \partial x_j^\ell}\right|_p f_2 = \frac{1}{2}\sum_{\alpha\le\beta} b_{\alpha\beta}^i \left(\frac{\partial x_i^\alpha}{\partial x_j^\ell}(p)\frac{\partial x_i^\beta}{\partial x_j^k}(p) + \frac{\partial x_i^\alpha}{\partial x_j^k}(p)\frac{\partial x_i^\beta}{\partial x_j^\ell}(p)\right)$$

both depend smoothly on $p$. Hence, $\psi_j \circ \psi_i^{-1}$ is smooth on $\psi_i(U_i \cap U_j)$ and we conclude that $E$ admits a smooth atlas. Furthermore, $E$ is countable since it has a countable atlas and to see that $E$ is Hausdorff let $(p, f_1, f_2)$ and $(q, g_1, g_2)$ be distinct points in $E$. If $p \ne q$ then there are open sets $p \in U$ and $q \in V$ in $M$ which separate $p$ and $q$ and $\pi^{-1}(U)$ and $\pi^{-1}(V)$ will separate $(p, f_1, f_2)$ and $(q, g_1, g_2)$. If $p = q \in U_i$ for some $i \in I$ then $(p, f_1, f_2)$ and $(q, g_1, g_2)$ can be separated in $\pi^{-1}(U_i)$ since the latter is homeomorphic to an open set in Euclidean space and therefore Hausdorff.

Exercise 3.4. Let $f\colon M \longrightarrow N$ be a smooth map between manifolds. Prove that the diagram

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{f^*} & \Omega^0(M) \\ \big\downarrow{\scriptstyle d} & & \big\downarrow{\scriptstyle d} \\ \Omega^1(N) & \xrightarrow[f^*]{} & \Omega^1(M) \end{array}$$

commutes.

*Solution.* Let $g \in \Omega^0(N) = C^\infty(N)$. Then $\mathrm{d}(f^*(g)) = \mathrm{d}(g \circ f)$ and $f^*(\mathrm{d}(g)) = f^*(\mathrm{d}g)$. To check that these coincide, let $x \in M$. We consider $T^*_x M$ to be $\mathscr{F}_x/\mathscr{F}_x^2$ and compute

$$\begin{aligned} \mathrm{d}(g \circ f)(x) &= [g \circ f - g(f(x))] \\ f^*(\mathrm{d}g)(x) &= f^*(\mathrm{d}g(f(x))) = f^*([g - g(f(x))]) = [(g - g(f(x))) \circ f]. \end{aligned}$$

But these are the same since $g(f(x))$ is just some real number.

Exercise 3.5. Give a detailed proof that the cotangent bundle $T^*M$ is a smooth manifold and that the projection map $\pi\colon T^*M \longrightarrow M$ is smooth.
*Solution.* Recall that

$$T^*M = \bigsqcup_{p \in M} T^*_p M$$

with the evident projection $\pi\colon T^*M \longrightarrow M$. Let $\{(U_i, \phi_i = (x_i^1, \dots, x_i^n))\}_{i \in I}$ be a countable smooth atlas for $M$. For each $i \in I$ define a map $\psi_i\colon \pi^{-1}(U_i) \longrightarrow \phi_i(U_i) \times \mathbb{R}^n$ by

$$\psi_i(p, \xi) = \Big(\phi_i(p), \big(\xi(\partial^i_\mu)\big)_\mu\Big).$$

Here $\partial^i_\mu$ is a convenient shorthand for $\partial/\partial x_i^\mu$. There is a unique topology on $T^*M$ making the $\psi_i$ into homeomorphisms. In this way we obtain an atlas for $T^*M$. We check that the transition maps are smooth. Assume $x = (\phi_i(p), (\xi^i_\mu)_\mu) \in \psi_i(U_i \cap U_j)$. Then

$$\psi_j(\psi_i^{-1}(x)) = \psi_j\Big(p, \sum_\mu \xi^i_\mu \, \mathrm{d}x_i^\mu\Big) = \Big(\phi_j(p), \Big(\sum_\mu \xi^i_\mu \, \mathrm{d}x_i^\mu(\partial^j_\nu)\Big)_\nu\Big) = \Big(\phi_j(p), \Big(\sum_\mu \xi^i_\mu \frac{\partial x_i^\mu}{\partial x_j^\nu}(p)\Big)\Big)$$

depends smoothly on $x$.

Since we were able to exhibit a countable atlas for $T^*M$ the cotangent bundle is second countable. To check that it is Hausdorff let $(p, \xi)$ and $(q, \zeta)$ be distinct points in $T^*M$. If $p \neq q$ then there are disjoint open sets $U$ and $V$ in $M$ containing $p$ and $q$ respectively and $\pi^{-1}(U)$ and $\pi^{-1}(V)$ separate $(p, \xi)$ and $(q, \zeta)$. If $p = q$ then there is some $U_i$ containing $p$ and then $(p, \xi)$ and $(q, \zeta)$ lie both inside $\pi^{-1}(U_i)$. The latter is Hausdorff, so we can separate $(p, \xi)$ and $(q, \zeta)$.

Exercise 3.6. Let $f$ and $g$ be smooth real-valued functions on a manifold $M$. Prove that $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$.
*Solution.* Recall that $\mathrm{d}h(p) = [h - h(p)] \in \mathscr{F}_p/\mathscr{F}_p^2$ for every $p \in M$ and $h\colon M \longrightarrow \mathbb{R}$ a smooth function. We compute

$$\begin{aligned} \mathrm{d}(fg)(p) &= [fg - f(p)g(p)] = [fg - (f - f(p))(g - g(p)) - f(p)g(p)] = \\ &= [f(p)g - f(p)g(p) + g(p)f - g(p)f(p)] = f(p)[g - g(p)] + g(p)[f - f(p)] = \\ &= f(p)\,\mathrm{d}g(p) + g(p)\,\mathrm{d}f(p) \end{aligned}$$

for every $p \in M$. We conclude that $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$.

Exercise 3.7. Let $i\colon S^1 = [0, 2\pi]/\{0, 2\pi\} \longrightarrow \mathbb{R}^2$ be the map $\theta \longmapsto (\cos(\theta), \sin(\theta))$. Compute the differential form $i^*((x^2 + y)\,\mathrm{d}x + (3 + xy^2)\,\mathrm{d}y)$.
*Solution.* We first describe a convenient basis of $T^*_p S^1$ for every $p \in S^1$. Let $\psi\colon (0, 2\pi) \longrightarrow (0, 2\pi)$ be the identity. For every $p \in (0, 2\pi) \subset S^1$ this map gives a nonzero covector $\mathrm{d}\psi(p) = [\psi - p] \in \mathscr{F}_p/\mathscr{F}_p^2$; in fact we get a 1–form $\mathrm{d}\psi$ defined on $(0, 2\pi)$ in this way. Let $\phi\colon [0, \pi) \cup (\pi, 2\pi] \longrightarrow \mathbb{R}$ be defined by $\phi(p) = p$ for $p \in [0, \pi)$ and $\phi(p) = p - 2\pi$ for $p \in (\pi, 2\pi]$. Then $\phi$ descends to a function on an open subset of $S^1$ and for $p \in (0, \pi)$ we have $\mathrm{d}\psi(p) = \mathrm{d}\phi(p)$. For $p \in (\pi, 2\pi)$ we find $\mathrm{d}\phi(p) = [\phi - p - 2\pi] = [\psi - p] = \mathrm{d}\psi(p)$.

We conclude that we have a globally defined 1–form $\mathrm{d}\theta \in \Omega^1(S^1)$ such that $\mathrm{d}\theta(p)$ generates $T_p^* S^1$ for every $p \in S^1$ and $\mathrm{d}\theta|_{(0,2\pi)} = \mathrm{d}\psi$ while $\mathrm{d}\theta|_{[0,\pi)\cup(\pi,2\pi]} = \mathrm{d}\phi$.

We prove a version of the chain rule for the exterior derivate d. If $f\colon M \longrightarrow \mathbb{R}$ and $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ are smooth functions and $p \in M$, then expand $g$ around $f(p)$ as

$$g(x) = g(f(p)) + g'(f(p))(x - f(p)) + h(x)(x - f(p))^2$$

for some smooth function $h\colon \mathbb{R} \longrightarrow \mathbb{R}$. Then

$$\begin{aligned} \mathrm{d}(g \circ f)(p) &= [g \circ f - g(f(p))] = [g(f(p)) + g'(f(p))(f - f(p)) + (h \circ f) \cdot (f - f(p))^2 - g(f(p))] = \\ &= g'(f(p))[f - f(p)] = g'(f(p))\, \mathrm{d}f(p). \end{aligned}$$

Therefore we conclude that $\mathrm{d}(g \circ f) = g'\, \mathrm{d}f$ as expected.

Let $\omega = (x^2 + y)\, \mathrm{d}x + (3 + xy^2)\, \mathrm{d}y$. For $p \in S^1$ we compute

$$\begin{aligned} i^*\omega(p) &= (x(i(p))^2 + y(i(p)))\, \mathrm{d}(x \circ i) + (3 + x(i(p))y(i(p))^2)\, \mathrm{d}(y \circ i) = \\ &= (\cos^2\theta(p) + \sin\theta(p))\, \mathrm{d}(\cos\theta) + (3 + \cos\theta(p)\sin^2\theta(p))\, \mathrm{d}(\sin\theta) = \\ &= (-\cos^2\theta(p)\sin\theta(p) - \sin^2\theta(p) + 3\cos\theta(p) + \cos^2\theta(p)\sin^2\theta(p))\, \mathrm{d}\theta \end{aligned}$$

and therefore

$$i^*\omega = (\cos^2\theta \sin^2\theta + 3\cos\theta - \sin^2\theta - \cos^2\theta\sin\theta)\, \mathrm{d}\theta.$$

EXERCISE 3.8. Earlier in class we defined the notion of a *category* $\mathscr{C}$; examples given include *topological spaces* Top and *vector spaces* Vect.

(i) Attached to any topological space $M$, define a category $\mathrm{Open}(M)$ as follows. Objects of $\mathrm{Open}(M)$ are the open subset $U \subset M$. Morphisms from $U$ to $V$ are *inclusions*, meaning that: if $U$ is not contained in $V$, then $\mathrm{Hom}(U, V) = \emptyset$ and if $U \subset V$, then $\mathrm{Hom}(U, V) = \{\iota_{UV}\}$ where $\iota_{UV}\colon U \hookrightarrow V$ is the inclusion map. Composition of morphisms is the usual composition of inclusions.
Verify that $\mathrm{Open}(M)$ satisfies the axioms of a category.

(ii) A *presheaf* on $M$ taking values in a category $\mathscr{C}$ is a functor $F\colon \mathrm{Open}(M)^{\mathrm{op}} \longrightarrow \mathscr{C}$. For instance, if $\mathrm{Alg}_{\mathbb{R}}$ denotes the category of $\mathbb{R}$–algebras with morphisms the $\mathbb{R}$–algebra homomorphisms, then a *presheaf of $\mathbb{R}$–algebras* on $M$ is a functor $F\colon \mathrm{Open}(M)^{\mathrm{op}} \longrightarrow \mathrm{Alg}_{\mathbb{R}}$.
Let $M$ be a smooth manifold now, and define $C^\infty(\_)\colon \mathrm{Open}(M)^{\mathrm{op}} \longrightarrow \mathrm{Alg}_{\mathbb{R}}$ by, on objects

$$U \longmapsto C^\infty(U)$$

and on the inclusions $\iota_{UV}\colon U \longrightarrow V$ the induced map $C^\infty(\_)_{UV}(\iota_{UV}) \in \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{R}}}(C^\infty(V), C^\infty(U))$ is the restriction map on functions $\iota_{UV}^*\colon C^\infty(V) \longrightarrow C^\infty(U)$.
Verify that $C^\infty(\_)$ is indeed a presheaf of $\mathbb{R}$–algebras and in particular a contravariant functor.

(iii) Verify that the notion of a presheaf of algebras $\mathscr{F}$ is equivalent to the following data:
- For every open set $U \in M$ an algebra $\mathscr{F}(U)$.
- For every inclusion of open sets $U \subset V$ a restriction map $\rho_{U\subset V}\colon \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$ satisfying, for any triple $U \subset V \subset W$, that $\rho_{U\subset V} \circ \rho_{V\subset W} = \rho_{U\subset W}$.

(iv) A presheaf as defined in the previous section is said to be a *sheaf* if for any pair of open sets $U$ and $V$, whenever there is an element $f_1 \in \mathscr{F}(U)$ and an element $f_2 \in \mathscr{F}(V)$ with the same restriction on the overlapping region $U \cap V$, then there exists a unique element $g \in \mathscr{F}(U \cup V)$ restricting to $f_1$ and $f_2$ on $U$ and $V$ respectively.
Let $M$ be a manifold. Verify that the presheaf $C^\infty(\_)$ on $M$ defined above is in fact a sheaf.

*Solution.* To check $\mathsf{Open}(M)$ is indeed a category, first note that the composition operation is just composition of functions and therefore is associative. For $U \in \mathsf{Open}(M)$ the identity function is the inclusion $U \hookrightarrow U$ and, again, because composition in $\mathsf{Open}(U)$ is composition of functions, this is an identity morphism.

To verify that $C^\infty(\_)$ is a contravariant functor, let $U \subset V \subset W$ be open sets in $M$ and let $f \in C^\infty(W)$. Then

$$\iota^*_{UW}(f)(x) = f(\iota_{UW}(x)) = f(x) = f(\iota_{VU}(\iota_{WV}(x))) = (\iota^*_{VU}(\iota^*_{WU}f))(x)$$

for all $x \in U$ and therefore $\iota^*_{UW}(f) = \iota^*_{VU}(\iota^*_{WU}(f))$ and $\iota^*_{UW} = \iota^*_{VU} \circ \iota^*_{WU}$ as required.

Since the only morphisms in $\mathsf{Open}(M)$ are the inclusions of open subsets, the only data needed to define a presheaf on morphisms is the image of $\iota_{UV}$ for open sets $U \subset V$. The compatibility condition on the presheaf is precisely the given condition on the restriction maps.

Take open sets $U, V \subset M$ and functions $f_1 \in C^\infty(U)$ and $f_2 \in C^\infty(V)$. Assume $\iota^*_{U\cap V,U}(f_1) = \iota^*_{U\cap V,V}(f_2)$. This just means that $f_1(x) = f_2(x)$ for all $x \in U \cap V$. Define a function $g\colon U \cup V \longrightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} f_1(x) & x \in U \\ f_2(x) & x \in V. \end{cases}$$

Since $f_1(x) = f_2(x)$ for $x \in U \cap V$ this definition makes sense. To check that $g \in C^\infty(U \cup V)$ note that a function is smooth if and only if it is smooth at every point in its domain. But $g|_U = f_1$ and $g|_V = f_2$ are both smooth, so $g$ is smooth at every point in $U \cup V$.

If $h$ is any other function on $U \cup V$ with $\iota^*_{U,U\cup V}(h) = f_1$ and $\iota^*_{V,U\cup V}(h) = f_2$, then we have $h(x) = f_1(x)$ for $x \in U$ and $h(x) = f_2(x)$ for $x \in V$. That is, $h(x) = g(x)$ for all $x \in U \cup V$, so that $h = g$.