Math 535a Homework 4

Due Friday, February 24, 2017 by 5 pm

Please remember to write down your name on your assignment.

1. In $\mathbb{R}^2$, consider the vector fields X and Y defined by
$$X = e^{x^2+y^2}\frac{\partial}{\partial x} + \sin(xy)\frac{\partial}{\partial y}$$
$$Y = (x^2+3xy)\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}.$$
Compute the Lie bracket $[X, Y]$.

2. Consider the vector field $X = x^2\frac{d}{dx}$ on $\mathbb{R}$. Compute its integral curves. Explain why X does not admit a global flow $\Phi : \mathbb{R} \times (-\epsilon, \epsilon) \to \mathbb{R}$ for any ϵ.

3. Let $\mathcal{D} = \ker(dz + (xdy - ydx)) \subset T\mathbb{R}^3$ be the two-dimensional distribution considered in class, called the *(standard) contact distribution* on $\mathbb{R}^3$. Verify that $\mathcal{D}$ is not integrable.

4. A *Lie group* is a manifold G equipped with a multiplication map $G \times G \to G$ which both satisfies the axioms of a group, and is a C^∞ map, such that the map $G \to G$ sending $g \mapsto g^{-1}$ is also C^∞.

   For an element $g \in G$, let $L_g : G \to G$ be the left multiplication, defined by $L_g(h) = gh$. A vector field X on G is *left invariant* if $(L_g)_*(X_h) = X_{gh}$[1] for every $g, h \in G$.

   (a) Show that, if $\mathbf{1} \in G$ denotes the identity element of G, then the map $X \mapsto X_{\mathbf{1}}$ induces a linear isomorphism between the vector space of all left invariant vector fields and the tangent space $T_{\mathbf{1}}G$.

   (b) Suppose G is a group of matrices that is a submanifold of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ (for instance $G = GL_n(\mathbb{R}), SL_n(\mathbb{R}), O_n(\mathbb{R}), or SO_n(\mathbb{R})$). (You may take for granted in this exercise that such a group is in fact a Lie group). Let X and Y be two left invariant vector fields. Show that
   $$[X, Y]_{\mathbf{1}} = X_{\mathbf{1}}Y_{\mathbf{1}} - Y_{\mathbf{1}}X_{\mathbf{1}}$$
   where, on the right hand side, the product is just the usual multiplication of matrices in $M_{n\times n}(\mathbb{R})$.

5. (worth double weight) Write out a proof of the Frobenius theorem in the general case.

6. Let $f : M^m \to \mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^{N-p}$ be an embedding of an m-dimensional manifold into Euclidean space.

   (a) Show that every horizontal subspace $\mathbb{R}^p \times \{z_0\}$ is arbitrarily close to a subspace $\mathbb{R}^p \times \{z\}$ whose pre-image $f^{-1}(\mathbb{R}^p \times \{z\})$ is an $(m - N + p)$-dimensional submanifold

[1] If $f : M \to N$ is a map between smooth manifolds, then there is an induced map $f_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ given by $(f_*X)_p = X_{f(p)}$

of M. (note: there is a reasonable notion of $m-N+p$-dimensional manifold assuming $m-N+p \geq 0$, with the convention that $\mathbb{R}^0 = \{0\}$, so 0-manifolds are collections of points. In this problem, we allow $m-N+p$ to be arbitrary, with the convention that the only possible negative dimensional manifold is the empty set).

(b) Show that, for $z \in \mathbb{R}^{N-p}$ as in the previous part, the intersection $f(M) \cap \mathbb{R}^p \times \{z\}$ is a submanifold of $\mathbb{R}^N$.

7. Let $f : M^m \to \mathbb{R}^N$ be an immersion from a smooth manifold of dimension m to $\mathbb{R}^N$.

(a) Let $T^1M \subset TM$ be the locus

$$T^1M = \{(x, v) \in TM \mid ||df_x(v)|| = 1\}.$$

Show that T^1M is a smooth manifold of dimension $2m-1$.

(b) Show, adapting arguments given in class, that if if $N > 2m$, then there exists $v \in S^{N-1} = \{v \in \mathbb{R}^N | ||v|| = 1\}$ such that $\pi_v \circ f$ is still an immersion, where π_v is the orthogonal projection from $\mathbb{R}^N$ to $H_v = \{w \in \mathbb{R}^N | w \cdot v = 0\}$. Conclude that, at least if M is compact, that there exists an immersion $g : M \to \mathbb{R}^{2m}$.