Homework 4

EXERCISE 4.1. In $\mathbb{R}^2$, consider the vector field X and Y defined by

$$X = e^{x^2+y^2}\frac{\partial}{\partial x} + \sin(xy)\frac{\partial}{\partial y}$$
$$Y = (x^2+3xy)\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}.$$

Compute the Lie bracket $[X, Y]$.
Solution. We know from general theory that all higher order derivatives while computing $[X, Y]$ must cancel. So we compute while disregarding higher order terms:

$$\begin{aligned} XY &\equiv \left(e^{x^2+y^2}\frac{\partial}{\partial x} + \sin(xy)\frac{\partial}{\partial y}\right) Y \equiv \\ &\equiv e^{x^2+y^2}(2x+3y)\frac{\partial}{\partial x} + e^{x^2+y^2}\frac{\partial}{\partial y} + 3x\sin(xy)\frac{\partial}{\partial x} + \sin(xy)\frac{\partial}{\partial y} \equiv \\ &\equiv \left(e^{x^2+y^2}(2x+3y) + 2x\sin(xy)\right)\frac{\partial}{\partial x} + \left(e^{x^2+y^2} + \sin(xy)\right)\frac{\partial}{\partial y} \end{aligned}$$

and

$$\begin{aligned} YX &\equiv \left((x^2+3xy)\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}\right) X \equiv \\ &\equiv (x^2+3xy)2xe^{x^2+y^2}\frac{\partial}{\partial x} + (x^2+3xy)y\cos(xy)\frac{\partial}{\partial y} + (x+y)2ye^{x^2+y^2}\frac{\partial}{\partial x} + (x+y)x\cos(xy)\frac{\partial}{\partial y} \equiv \\ &\equiv e^{x^2+y^2}(2x(x^2+3xy) + 2y(x+y))\frac{\partial}{\partial x} + \cos(xy)((x^2+3xy)y + (x+y)x)\frac{\partial}{\partial y} \end{aligned}$$

and therefore

$$\begin{aligned} [X, Y] = &\left(e^{x^2+y^2}(-2x^3 - 6x^2y - 2xy + 2x - 2y^2 + 3y) + 2x\sin(xy)\right)\frac{\partial}{\partial x} + \\ &\left(e^{x^2+y^2} + \sin(xy) - \cos(xy)(x^2y + x^2 + 3xy^2 + xy)\right)\frac{\partial}{\partial y} \end{aligned}$$

EXERCISE 4.2. Consider the vector field $X = x^2\frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathbb{R}$. Compute its integral curves. Explain why X does not admit a global flow $\Phi\colon \mathbb{R}\times(-\varepsilon,\varepsilon) \longrightarrow \mathbb{R}$ for any $\varepsilon > 0$.
Solution. Assume $\gamma\colon (-\varepsilon,\varepsilon) \longrightarrow \mathbb{R}$ is an integral curve of X with $\gamma(0) = p \in \mathbb{R}$. Let ∂_t be the basis vector field for the tangent bundle of $(-\varepsilon,\varepsilon)$ and write ∂_x for the basis vector field of $T\mathbb{R}$. Then γ being a integral curve of X means

$$\gamma(t)^2\partial_{x,\gamma(t)} = X_{\gamma(t)} = \gamma_*(\partial_t)_t = \gamma'(t)\partial_{x,\gamma(t)}$$

for all t in the domain of γ. Hence γ satisfies the initial value problem

$$\gamma'(t) = \gamma(t)^2, \quad \gamma(0) = p.$$

This initial value problem is solved by $\gamma(t) = p/(1-pt)$ for $t < 1/p$ if $p \neq 0$ and all $t \in \mathbb{R}$ if $p = 0$.

If X were to admit a global flow $\Phi\colon \mathbb{R}\times(-\varepsilon,\varepsilon) \longrightarrow \mathbb{R}$, then we would necessarily have $\Phi(p,t) = p/(1-pt)$. We have $\Phi(p,t) \longrightarrow \infty$ for $t \longrightarrow 1/p$. Hence, Φ cannot be smooth at $(2/\varepsilon, \varepsilon/2) \in \mathbb{R}\times(-\varepsilon,\varepsilon)$, for instance.

EXERCISE 4.3. Let $\mathscr{D} = \ker(\mathrm{d}z + (x\,\mathrm{d}y - y\,\mathrm{d}x)) \subset T\mathbb{R}^3$ be the two-dimensional distribution considered in class, called the *(standard) contact distribution* on $\mathbb{R}^3$. Verify that $\mathscr{D}$ is not integrable.

Solution. By the Frobenius theorem it is enough to show that $\mathscr{D}$ is not involutive. Write $\omega = dz + (x\,dy - y\,dx)$ and observe that $\omega(\partial_x + y\partial_z) = y - y = 0$ and $\omega(\partial_y - x\partial_z) = x - x = 0$. We therefore conclude that $A = \partial_x + y\partial_z \in \mathscr{D}$ and $B = \partial_y - x\partial_z \in \mathscr{D}$. But

$$\begin{aligned}[A, B] &= [\partial_x + y\partial_z, \partial_y - x\partial_z] = [\partial_x, -x\partial_z] + [y\partial_z, \partial_y] + [y\partial_z, -x\partial_z] = \\ &= -\partial_z - \partial_z + 0 = -2\partial_z\end{aligned}$$

and $\omega(-2\partial_z) = -2$, so $-2\partial_z \notin \mathscr{D}$. In this calculation we have used that coordinate vector fields commute and that the Lie bracket satisfies a derivation-type identity

$$[X, fY] = X(f)Y + f[X, Y]$$

for vector fields X and Y and a smooth function f.

Exercise 4.4. A *Lie group* is a manifold G equipped with a multiplication map $G \times G \longrightarrow G$ which both satisfies the axioms of a group, and is a C^∞ map, such that the map $G \longrightarrow G$ sending $g \longmapsto g^{-1}$ is also C^∞.

For an element $g \in G$, let $L_g \colon G \longrightarrow G$ be the left multiplication, defined by $L_g(h) = gh$. A vector field X on G is *left invariant* if $(L_g)_*(X_h) = X_{gh}$ for every $g, h \in G$.

(i) Show that, if $\mathbf{1} \in G$ denotes the identity element of G, then the map $X \longmapsto X_\mathbf{1}$ induces a linear isomorphism between the vector space of all left invariant vector fields and the tangent space $T_\mathbf{1}G$.

(ii) Suppose G is a group of matrices that is a submanifold of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ (for instance $G = \mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{O}_n(\mathbb{R})$, or $\mathrm{SO}_n(\mathbb{R})$). You may take for granted in this exercise that such a group is in fact a Lie group. Let X and Y be two left invariant vector fields. Show that

$$[X, Y]_\mathbf{1} = X_\mathbf{1} Y_\mathbf{1} - Y_\mathbf{1} X_\mathbf{1}$$

where, on the right hand side, the product is just the usual multiplication of matrices in $M_{n\times n}(\mathbb{R})$.

Solution. We first construct an inverse map to $X \longmapsto X_\mathbf{1}$. Let $v \in T_\mathbf{1}G$ be any tangent vector. Let $f \colon G \longrightarrow \mathbb{R}$ be a smooth function and let $g \in G$. Let $\gamma \colon (-\varepsilon, \varepsilon) \longrightarrow G$ be a smooth curve with $\gamma(0) = \mathbf{1}$ and $\gamma'(0) = v$. Then

$$\left.\frac{d}{dt}\right|_{t=0} f(L_g(\gamma(t))) = \left.\frac{d}{dt}\right|_{t=0} f(g\gamma(t))$$

depends smoothly on g, so we can define a smooth function $X_v(f)$ by

$$X_v(f)(g) = \left.\frac{d}{dt}\right|_{t=0} f(g\gamma(t)).$$

Observe that $X_v(f)(g) = v(f \circ L_g) = (L_{g,*}v)(f)$. Since $L_{g,*}v$ is a derivation for each g, this implies that X_v defines a smooth vector field on G which satisfies $X_{v,g} = L_{g,*}v \in T_gG$ for all $g \in G$. It follows that $L_{h,*}(X_v)_g = L_{h,*}(L_{g,*}v) = L_{hg,*}v = X_{v,hg}$, i. e. X_v is a left invariant vector field on G with $X_{v,\mathbf{1}} = L_{\mathbf{1},*}v = v$. Conversely, if X is left invariant, then $X_g = L_{g,*}X_\mathbf{1}$ by definition. We conclude that $X \longmapsto X_\mathbf{1}$ is an isomorphism.

Let $x^{ij} \colon G \longrightarrow \mathbb{R}$ be the canonical coordinate functions of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ restricted to G. The associated coordinate vector fields on $M_{n\times n}(\mathbb{R})$ will be denoted by ∂_{ij}. Let X and Y be left invariant vector fields on G and write

$$X_\mathbf{1} = \sum_{ij} a^{ij}\partial_{ij,\mathbf{1}} \quad \text{and} \quad Y_\mathbf{1} = \sum_{k\ell} b^{k\ell}\partial_{k\ell,\mathbf{1}}.$$

For a matrix $g = (g^{ij}) \in G$ we have

$$X_g(x^{ij}) = (L_{g,*}X)_g(x^{ij}) = X_\mathbf{1}(x^{ij} \circ L_g) = X_\mathbf{1}\Big(\sum_k g^{ik}x^{kj}\Big) = \sum_k g^{ik}a^{kj} = \sum_k x^{ik}(g)a^{kj}.$$

We conclude that

$$X(x^{ij}) = \sum_k x^{ik} a^{kj} \quad \text{and} \quad Y(x^{ij}) = \sum_k x^{ik} b^{kj}$$

for $g \in G$. Therefore

$$[X, Y](x^{ij}) = X\Big(Y(x^{ij})\Big) - Y\Big(X(x^{ij})\Big) = \sum_k X(b^{kj} x^{ik}) - \sum_k Y(a^{kj} x^{ik}) =$$
$$= \sum_{k\ell} x^{i\ell} a^{\ell k} b^{kj} - \sum_{k\ell} x^{i\ell} b^{\ell k} a^{kj}$$

and

$$[X, Y]_{\mathbf{1}}(x^{ij}) = \sum_k a^{ik} b^{kj} - \sum_k b^{ik} a^{kj} = (X_{\mathbf{1}} Y_{\mathbf{1}} - Y_{\mathbf{1}} X_{\mathbf{1}})(x^{ij}).$$

Exercise 4.5. Write out a proof of the Frobenius theorem in the general case.

Solution. First, let $\mathscr{D}$ be a rank k distribution of $\mathbb{R}^n$. Let $X_1, \dots, X_k$ be a local frame for $\mathscr{D}$ in an open neighborhood U of 0. Suppose first that $[X_i, X_j] = 0$ and let $\varepsilon > 0$ and $\delta > 0$ be small enough such that each X_i admits a flow $\varphi_i \colon (-\varepsilon, \varepsilon)^n \times (-\delta, \delta) \longrightarrow (-\varepsilon, \varepsilon)^n$. Define

$$\varphi_{i_1,\dots,i_k}(s^1, \dots, s^k) = \varphi_{i_1}(\_, s^1) \circ \dots \circ \varphi_{i_k}(\_, s^k)$$

for any permutation $(i_1, \dots, i_k)$ of $(1, \dots, k)$. Set

$$\Phi \colon (-\delta, \delta)^k \times (-\varepsilon, \varepsilon)^{n-k} \longrightarrow (-\varepsilon, \varepsilon)^n;\ (s^1, \dots, s^n) \longmapsto \varphi_{1\dots k}(s^1, \dots, s^k)(0, \dots, 0, s^{k+1}, \dots, s^n)$$

and compute

$$\begin{aligned} \mathrm{d}\Phi_p \left(\frac{\partial}{\partial s^i}\bigg|_p \right)(f) &= \frac{\partial}{\partial s^i}\bigg|_p f(\Phi(s^1, \dots, s^n)) = \\ &= \frac{\partial}{\partial s^i}\bigg|_p f(\varphi_{1\dots k}(s^1, \dots, s^k)(0, \dots, 0, s^{k+1}, \dots, s^n)) = \\ &= \frac{\partial}{\partial s^i}\bigg|_p f(\varphi_{i,1,\dots,\widehat{i},\dots,k}(s^1, \dots, s^k)(0, \dots, 0, s^{k+1}, \dots, s^n)) = \\ &= X_{i,\Phi(p)}(f) \end{aligned}$$

for all $p \in (-\delta, \delta)^k \times (-\varepsilon, \varepsilon)^{n-k}$ because the flows of the X_j commute. Since the vector fields X_j are linearly independent at 0 the differential $\mathrm{d}\Phi_0$ is an isomorphism. Therefore, for small enough neighborhoods V of 0, the restriction $\Phi|_V$ is a diffeomorphism $V \longrightarrow \Phi(V)$ and the image of $(-\delta, \delta)^k \times \{0\} \cap V$ is an integral submanifold of $\mathscr{D}$ near 0.

Next, if the X_i don't commute, let $\mathscr{D}_0 = \langle X_{1,0}, \dots, X_{k,0} \rangle \subset \mathbb{R}^n$ be the subspace spanned by $X_1, \dots, X_k$ at 0 and let Y be a subspace of $\mathbb{R}^n$ complementary to $\mathscr{D}_0$. Let $\pi \colon \mathbb{R}^n \longrightarrow \mathscr{D}_0$ be the linear projection with kernel Y. This induces a smooth bundle map $\mathrm{d}\pi \colon T\mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathscr{D}_0$. Observe that $(\mathrm{d}\pi|_{\mathscr{D}})_0 = \mathrm{d}\pi_0|_{\mathscr{D}_0}$ is bijective by construction. It follows that, for some open neighborhood $U \subset \mathbb{R}^n$ of 0, the restriction $\mathrm{d}\pi|_{\mathscr{D}} \colon \mathscr{D}|_U \longrightarrow U \times \mathscr{D}_0$ is a vector bundle isomorphism. Consider $X_{i,0}$ as constant section of $U \times \mathscr{D}_0$. Then, setting $V_{i,p} := \mathrm{d}\pi_p^{-1}(X_{i,0})$ defines smooth vector fields V_i on U which form a frame for $\mathscr{D}|_U$. By the naturality of the Lie bracket we have

$$\mathrm{d}\pi_q([V_i, V_j]_q) = [X_{i,0}, X_{j,0}]_{\pi(q)} = 0$$

because the $X_{i,0}$ are constant. But if $\mathscr{D}$ is involutive, $[V_i, V_j]_q \in \mathscr{D}_q$ and $\mathrm{d}\pi_q$ is injective on $\mathscr{D}_q$ for $q \in U$. Therefore $[V_i, V_j] = 0$ on U and we conclude that $\mathscr{D}$ is integrable at 0 by our first argument. The case of general manifolds follows by taking charts.

EXERCISE 4.6. Let $f\colon M^m \longrightarrow \mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^{N-p}$ be an embedding of an m–dimensional manifold into Euclidean space.

(i) Show that every horizontal subspace $\mathbb{R}^p \times \{z_0\}$ is arbitrarily close to a subspace $\mathbb{R}^p \times \{z\}$ whose preimage $f^{-1}(\mathbb{R}^p \times \{z\})$ is an $(m-N+p)$–dimensional submanifold of M.

(ii) Show that, for $z \in \mathbb{R}^{N-p}$ as in the previous part, the intersection $f(M) \cap \mathbb{R}^p \times \{z\}$ is a submanifold of $\mathbb{R}^N$.

Solution. Consider the orthogonal projection $\pi\colon \mathbb{R}^N \longrightarrow \mathbb{R}^{N-p}$ and observe that $\pi \circ f\colon M^m \longrightarrow \mathbb{R}^{N-p}$ is smooth. By Sard's theorem the set of regular values of $\pi\circ f$ is dense and therefore any $z_0 \in \mathbb{R}^{N-p}$ is arbitrarily close to a regular value $z \in \mathbb{R}^{N-p}$. But then $(\pi \circ f)^{-1}(z) = f^{-1}(\mathbb{R}^p \times \{z\})$ is an $(m-N+p)$–dimensional submanifold of M.

If f is an embedding, then $f(f^{-1}(\mathbb{R}^p \times \{z\})) = f(M) \cap \mathbb{R}^p \times \{z\}$ is a submanifold of $\mathbb{R}^N$ as well.

EXERCISE 4.7. Let $f\colon M^m \longrightarrow \mathbb{R}^N$ be an immersion from a smooth manifold of dimension m to $\mathbb{R}^N$.

(i) Let $T^1M \subset TM$ be the locus

$$T^1M = \{(x, v) \in TM : \|\mathrm{d}f_x(v)\| = 1\}.$$

Show that T^1M is a smooth manifold of dimension $2m-1$.

(ii) Show, adapting arguments given in class, that if $N > 2m$, then there exists $v \in S^{N-1}$ such that $\pi_v \circ f$ is still an immersion, where π_v is the orthogonal projection from $\mathbb{R}^N$ to $H_v = \{w \in \mathbb{R}^N : \langle w, v\rangle = 0\}$. Conclude that, at least if M is compact, that there exists an immersion $g\colon M \longrightarrow \mathbb{R}^{2m}$.

Solution. Consider the smooth map $\psi\colon TM \longrightarrow \mathbb{R}$ given by $\psi(x, v) = \|\mathrm{d}f_x(v)\|^2$. Let $p \in M$ and let $\varphi\colon U \longrightarrow \mathbb{R}^m$ be a chart centered at p in M. Then $\mathrm{d}\varphi\colon TU \longrightarrow \mathbb{R}^m \times \mathbb{R}^m$ is an isomorphism of vector bundles over U. The map $\psi \circ \mathrm{d}\varphi^{-1}\colon \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is given by

$$\Psi(\mathrm{d}\varphi^{-1}(x, v)) = \|\mathrm{d}(f \circ \varphi^{-1})_x(v)\|^2$$

and its derivative can be computed as

$$\mathrm{d}(\Psi \circ \mathrm{d}\varphi^{-1})_{(x,v)}(\xi, \zeta) = 2\left\langle \mathrm{d}(f\circ\varphi^{-1})_x(v), \mathrm{d}^2(f\circ\varphi^{-1})_x(v, \xi) + \mathrm{d}(f\circ\varphi^{-1})_x(\zeta)\right\rangle$$

for $(\xi, \zeta) \in \mathbb{R}^m \times \mathbb{R}^m$, where $\mathrm{d}^2(f\circ\varphi^{-1})_x(v, \xi)$ is the vector in $\mathbb{R}^N$ with components

$$\left(\mathrm{d}^2(f\circ\varphi^{-1})_x(v, \xi)\right)^i = \sum_{j,k} \frac{\partial^2 (f\circ\varphi^{-1})^i(x)}{\partial x^j \partial x^k} v^k \xi^j.$$

If $\|\mathrm{d}(f\circ\varphi^{-1})_x(v)\|^2 = 1$, then

$$\mathrm{d}(\Psi \circ \mathrm{d}\varphi^{-1})_{(x,v)}(0, v) = 2\left\langle \mathrm{d}(f\circ\varphi^{-1})_x(v), \mathrm{d}^2(f\circ\varphi^{-1})_x(v, 0) + \mathrm{d}(f\circ\varphi^{-1})_x(v)\right\rangle = 2$$

and we can conclude that $\mathrm{d}(\Psi \circ \mathrm{d}\varphi^{-1})_{(x,v)}\colon \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is a surjective linear map. Therefore 1 is a regular value of $\Psi \circ \mathrm{d}\varphi^{-1}$. By varying the chart ϕ and $p \in M$ it follows that 1 is also a regular value of Ψ. This makes $T^1M = \Psi^{-1}(1)$ a $(2m-1)$–dimensional submanifold of TM.

If $N > 2m$, denote the projection $T\mathbb{R}^N \longrightarrow \mathbb{R}^N$ mapping (x, ξ) to ξ by π and let $v \in S^{N-1}$ be a regular value of $\pi \circ \mathrm{d}f|_{T^1M}\colon T^1M \longrightarrow S^{N-1}$. Such a v exists by Sard's theorem. Because of the inequality $\dim(T^1M) = 2m-1 < N-1 = \dim(S^{N-1})$ this means that there is no $(x, \xi) \in T^1M$ with $\mathrm{d}f_x(\xi) = v$. We check that $\pi_v \circ f$ is an immersion of M into $\mathbb{R}^{N-1}$. Because π_v is linear we can compute

$$\mathrm{d}(\pi_v \circ f)_x(\xi) = \pi_v(\mathrm{d}f_x(\xi)) \in H_v$$

for $(x, \xi) \in TM$. Now, if $\mathrm{d}(\pi_v \circ f)_x(\xi) = 0$ for $\xi \neq 0$, then $\mathrm{d}f_x(\xi) \neq 0$ because f is an immersion and we must have $\mathrm{d}f_x(\xi) \parallel v$. But our choice of $v \in S^{N-1}$ ensures that this is impossible. Hence, $\mathrm{d}(\pi_v \circ f)_x$ is injective for all $x \in M$, i. e. $\pi_v \circ f\colon M \longrightarrow H_v \cong \mathbb{R}^{N-1}$ is an immersion. Repeating this procedure yields an immersion $M \longrightarrow \mathbb{R}^{2m}$.