Math 535a Homework 5

Due Monday, March 20, 2017 by 5 pm

Please remember to write down your name on your assignment.

1. Let (E, π_E) and (F, π_F) be (smooth) vector bundles over a common base M. A *vector bundle morphism over* M from E to F is a C^∞ map $f : E \to F$, compatible with projection, meaning that $\pi_F \circ f = \pi_E$, so that on each fiber $f_p : E_p \to F_p$ is a linear map of vector spaces. An *isomorphism of vector bundles* $E \cong F$ over potentially different manifolds M and N is a diffeomorphism $\bar{f} : M \to N$ and a vector bundle morphism covering $\bar{f}$ which is an isomorphism on each fiber.[1] We say E and F are *isomorphic over* M if the isomorphism of vector bundles covers the identity map; e.g, if the vector bundle morphism $f : E \to F$ satisfies $\pi_F \circ f = \pi_E$, so f maps E_p to F_p.[2]

   Let $(E, \pi : E \to M)$ be a rank k vector bundle. We say a collection of sections $s_1, \ldots, s_k \in \Gamma(E)$ is *linearly independent* if $(s_1)(p), \ldots, (s_k)(p)$ are linearly independent in E_p for each $p \in M$ (in particular, no $s_i(p)$ should be zero). Prove that there is an isomorphism of vector bundles over M, $E \cong \underline{\mathbb{R}}^k$ if and only if E has a basis of linearly independent sections $s_1, \ldots, s_k$. (**Recall**: $\underline{\mathbb{R}}^k$ denotes the trivial rank k bundle over M, defined as $\underline{\mathbb{R}}^k := M \times \mathbb{R}^k$, with projection map $\pi : M \times \mathbb{R}^k \to M$ sending (p, v) to p).

2. (double weight) **Vector bundles via gluing data**. Let M be a smooth manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open cover of M. A set of *transition* (or *gluing*) *data* (of rank k) for the cover $\mathcal{U}$ is a collection of C^∞ maps

   $$\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}) \text{ for all } \alpha, \beta \in I$$

   satisfying

   i. For any $\alpha, \beta \in I$, $\Phi_{\alpha\beta} \cdot \Phi_{\beta\alpha} = I$.[3]
   ii. (*cocycle condition*) for any $\alpha, \beta, \gamma \in I$, $\Phi_{\gamma\alpha} = \Phi_{\gamma\beta} \cdot \Phi_{\beta\alpha}$ as functions on $U_\alpha \cap U_\beta \cap U_\gamma$.

   (a) Show, as asserted in class, that, given a collection of transition data $\mathcal{T} := \{\Phi_{\alpha\beta}\}_{\alpha,\beta \in I}$ for the cover $\mathcal{U}$, there is a C^∞ rank k vector bundle $\mathcal{E}_{\mathcal{U},\mathcal{T}}$ over M, formed by gluing

[1] If $\bar{f} : M \to N$ is a C^∞ map, and $E, \pi_E : E \to M$, $F : \pi_F : F \to N$ are vector bundles, a *(general) bundle morphism covering* $\bar{f}$ is a C^∞ map $f : E \to F$ satisfying $\pi_F \circ f = \bar{f} \circ \pi_E$, so that the induced map on fibers $E_p \to F_{\bar{f}(p)}$ is a linear map.

[2] An implicit point is that isomorphism of bundles over M is an equivalence relation. In particular, you should check that if there is an isomorphism of vector bundles $f : E \xrightarrow{\sim} F$, then f is a diffeomorphism with $f^{-1} : F \to E$ also an isomorphism of vector bundles. You do not need to prove this fact as part of your HW, but if would like to use this fact on the HW elsewhere, you should prove it as a Lemma.

[3] Here and in ii., multiplication means pointwise multiplication of matrices, i.e., for $f, g : V \to GL(k, \mathbb{R})$, $(f \cdot g)(p) = f(p) \cdot g(p)$. Also, I means the constant function with value the identity matrix; $V \to GL(k, \mathbb{R})$, $p \mapsto I$.

together trivial vector bundles $U_\alpha \times \mathbb{R}^k$ via the transition data maps:

$$\mathcal{E}_{\mathcal{U},\mathcal{T}} := \coprod_{U_\alpha \in \mathcal{U}} (U_\alpha \times \mathbb{R}^k)/\{(U_\alpha, p, v) \sim (U_\beta, p, \Phi_{\alpha\beta}(p)(v)) \text{ for all } \alpha, \beta, p \in U_\alpha \cap U_\beta\}.$$

with projection map to M defined as follows: first, define

$$M_{\mathcal{U}} = \coprod_{U_\alpha \in \mathcal{U}} U_\alpha/\{(U_\alpha, p) \sim (U_\beta, p) \text{ for all } \alpha, \beta, p \in U_\alpha \cap U_\beta\};$$

check that $M_{\mathcal{U}}$ is indeed a smooth manifold, and show that there is a canonical diffeomorphism

$$i_{\mathcal{U}} : M_{\mathcal{U}} \xrightarrow{\sim} M$$

sending $[(U_\alpha, p)]$ to p. Then one defines $\pi : \mathcal{E}_{\mathcal{U},\mathcal{T}} \to M$ as the composition of the map $\pi_{\mathcal{T}} : \mathcal{E}_{\mathcal{U},\mathcal{T}} \to M_{\mathcal{U}}$ with $i_{\mathcal{U}}$, where $\pi_{\mathcal{T}} : [(U_\alpha, p, v)] \mapsto [(U_\alpha, p)]$ (again, you'll need to check this is C^∞).[4]

Note: part of this problem involves verifying that (a) of $M_{\mathcal{U}}$ as above is indeed a smooth manifold canonically diffeomorphic to M, and (b) $\mathcal{E}_{\mathcal{U},\mathcal{T}}$ has the structure of a smooth manifold with π a smooth map, and finally that (c) $(\mathcal{E}_{\mathcal{U},\mathcal{T}}, \pi)$ satisfy the axioms of a vector bundle.

(b) Show that any vector bundle E on M is isomorphic, as vector bundles over M (in the sense of problem 1), to $\mathcal{E}_{\mathcal{U},\mathcal{T}}$ for some cover $\mathcal{U}$ and some transition data $\mathcal{T}$.

(c) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover and $\mathcal{T} = \{\Phi_{\alpha\beta}\}$ a set of transition data of rank k. A *(Čech) 0-co-chain* on $\mathcal{U}$ is a collection of functions $\mathcal{F} := \{f_\alpha : U_\alpha \to GL(k, \mathbb{R})\}$. Given such a Čech 0-co-chain, define new transition data for the cover $\mathcal{U}$ by

$$\mathcal{F}\mathcal{T} := \{f_\alpha^{-1} \cdot \Phi_{\alpha\beta} \cdot f_\beta\}.$$

Show that $\mathcal{E}_{\mathcal{U},\mathcal{T}}$ and $\mathcal{E}_{\mathcal{U},\mathcal{F}\mathcal{T}}$ are isomorphic as vector bundles over M.

(d) Given an open cover of M, $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$, along with transition data $\mathcal{T}$ for $\mathcal{U}$, a *refinement* of the cover $\mathcal{U}$ is a new open cover $\mathcal{U}' = \{V_\kappa\}_{\kappa \in J}$ along with a map of index sets $f : J \to I$ such that for each $\kappa \in J$,

$$V_\kappa \subset U_{f(\kappa)}.$$

(**Example**: If $M = \mathbb{R}^n$, one can take $\mathcal{U} = \{\mathbb{R}^n\}$ and $\mathcal{U}' = \{B_\epsilon(p)\}_{p \in J = \mathbb{R}^n}$. Then $\mathcal{U}'$, equipped with the canonical map on index sets $f : J \to \{*\}$, is a refinement of $\mathcal{U}$).

Given a refinement of $\mathcal{U}$, call it $\mathcal{U}'$, the *induced transition data on the refinement* is

$$\mathcal{T}|_{\mathcal{U}'} := \{\Phi_{f(\kappa)f(\tau)}|_{V_\kappa \cap V_\tau}\}_{\kappa, \tau \in J}.$$

Show that there is an isomorphism of vector bundles over M, $\mathcal{E}_{\mathcal{U}', \mathcal{T}|_{\mathcal{U}'}} \cong \mathcal{E}_{\mathcal{U},\mathcal{T}}$.

[4] **Note**: In this entire problem, for clarity we're using the notation $\coprod_{V \in \mathcal{V}} V := \{(V, p) | V \in \mathcal{V}, p \in V\}$. So elements of disjoint unions should be thought of as tuples (V, p) where V is one of the sets we're taking the disjoint union of and p is an element of V.

(e) Let $Vect^k(M)$ denote the set of smooth rank k vector bundles on M up to isomorphism. That is,

$$Vect^k(M) := \{(E, \pi) \text{ a vector bundle over } M\}/\sim,$$

where $E \sim F$ if E and F are isomorphic over M, in the terminology of problem 1. Show that there is a bijection of sets

$$Vect^k(M) \cong \{(\mathcal{U}, \mathcal{T}) \mid \mathcal{U} \text{ is any open cover of } M \text{ and } \mathcal{T} \text{ is any transition data of rank } k \text{ for } \mathcal{U}\}/\sim,$$

where $(\mathcal{U}_1, \mathcal{T}_1) \sim (\mathcal{U}_2, \mathcal{T}_2)$ if, after passing to a common refinement $\mathcal{U}'$ of $\mathcal{U}_1$ and $\mathcal{U}_2$, there is a Čech 0-co-chain $\mathcal{F}$ on $\mathcal{U}'$ such that

$$\mathcal{T}_1|_{\mathcal{U}'} = \mathcal{F}(\mathcal{T}_2|_{\mathcal{U}'}).$$

Hint: there is a natural map $\{(\mathcal{U}, \mathcal{T}) \text{ as above}\} \to Vect^k(M)$ sending $(\mathcal{U}, \mathcal{T})$ to $(\mathcal{E}_{\mathcal{U},\mathcal{T}}, \pi)$. Show that this descends to a well-defined map $\{(\mathcal{U}, \mathcal{T}) \text{ as above}\}/\sim \to Vect^k(M)$ which is in fact a bijection.

3. (double weight) **The category theory of vector bundle operations**: In class, we have indicated or sketched how many vector space operations, such as direct sum, tensor product, self tensor/exterior product, quotient, etc. induce operations on vector bundles. In this homework assignment we will give a streamlined proof of all such constructions, using some category theory.

First, if $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are categories, the *product category* $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ has as objects elements of the Cartesian product $(\text{ob } \mathcal{C}_1) \times \cdots \times (\text{ob } \mathcal{C}_k)$, and has morphisms, for a pair of tuples $(X_1, \ldots, X_k)$, $(Y_1, \ldots, Y_k)$,

$$\hom_{\mathcal{C}_1 \times \cdots \times \mathcal{C}_k}((X_1, \ldots, X_k), (Y_1, \ldots, Y_k)) := \hom_{\mathcal{C}_1}(X_1, Y_1) \times \cdots \times \hom_{\mathcal{C}_k}(X_k, Y_k),$$

with composition given by component-wise composition (see exercise (a) below). We use the shorthand

$$\mathcal{C}^k := \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{k \text{ times}}$$

A *k-ary functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor

$$F : \mathcal{C}^k \to \mathcal{D}.$$

Now, let

$$\mathcal{V} := Vect_{\mathbb{R}}^{iso}$$

be the category whose objects are finite-dimensional $\mathbb{R}$-linear vector spaces and whose morphisms are the $\mathbb{R}$-linear isomorphisms/invertible linear transformations, with composition given by composition of linear maps (you should convince yourself this is a category). Note that in this category $\hom_{\mathcal{V}}(V, V) = GL(V)$.

(a) Check that the product category $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ is indeed a category if $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are.

(b) Show that in $\mathcal{V}$, $\hom_{\mathcal{V}}(V, W)$ has the structure of a C^∞ manifold for any pair of finite-dimensional vector spaces V, W. Moreover, check that the composition map $\hom_{\mathcal{V}}(W, Z) \times \hom_{\mathcal{V}}(V, W) \to \hom_{\mathcal{V}}(V, Z)$ is a C^∞ map of manifolds. (we have already seen this when $V = W = Z$) (in categorical language, we might say that $\mathcal{V}$ is "enriched in the category of smooth manifolds.")

(c) We say a k-ary functor $T : \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{k \text{ times}} \to \mathcal{V}$ is *smooth* if, for any pair of objects $(V_1, \ldots, V_k), (W_1, \ldots, W_k)$ in $\mathcal{V}^k$, the the induced map on morphism spaces

$$\begin{aligned} \hom_{\mathcal{V}}(V_1, W_1) \times \cdots \times \hom_{\mathcal{V}}(V_k, W_k) &= \hom_{\mathcal{V}^k}((V_1, \ldots, V_k), (V_1, \ldots, V_k)) \\ &\to \hom_{\mathcal{V}}(T(V_1, \ldots, V_k), T(W_1, \ldots, W_k)) \end{aligned}$$

is a C^∞ map of manifolds.

Show that the following give smooth k-ary functors on $\mathcal{V}$. Note: you will in particular need to define a map on morphism spaces associated to the map on objects we define below.

- 2-ary functor $T_\oplus$, defined on objects by $T_\oplus : (V, W) \mapsto V \oplus W$.
- 2-ary functor $T_\otimes$, defined on objects by $T_\otimes : (V, W) \mapsto V \otimes W$.
- 1-ary functor $T_{\otimes k}$, defined on objects by $T_{\otimes k} : V \mapsto V^{\otimes k}$ for some k.
- 1-ary functor $T_{\wedge k}$, defined on objects by $T_{\wedge k} : V \mapsto \wedge^k V$ for some k.

(d) Given a tuple of (smooth) vector bundles $\mathcal{E}_1, \ldots, \mathcal{E}_k$ over a manifold M, and a smooth k-ary functor T as above, directly construct a smooth vector bundle

$$\underline{T}(\mathcal{E}_1, \ldots, \mathcal{E}_k)$$

with $\underline{T}(\mathcal{E}_1, \ldots, \mathcal{E}_k)_p = T((\mathcal{E}_1)_p, \ldots, (\mathcal{E}_k)_p)$.

Hint: This problem is very similar to the problem of constructing TM. First, as a set $\underline{T}(\mathcal{E}_1, \ldots, \mathcal{E}_k) = \{(p, v) | p \in M, v \in T((\mathcal{E}_1)_p, \ldots, (\mathcal{E}_k)_p)\}$. Next, you will need to equip this set with a topology (use a cover on M consisting of open sets for which the input vector bundles all simultaneously admit trivializations), and manifold structure, such that the projection map to M is C^∞. Finally, you should argue that the resulting pair $(\underline{T}(\mathcal{E}_1, \ldots, \mathcal{E}_k), \pi)$ is indeed a smooth vector bundle.

(e) Give a construction of the vector bundle $\underline{T}(\mathcal{E}_1, \ldots, \mathcal{E}_k)$ in terms of transition data (as in the previous problem), using as input the transition data associated to $\mathcal{E}_1, \ldots, \mathcal{E}_k$ for some cover, and explain why the result is isomorphic (as vector bundles) to the construction above.

Note: Now, if $\mathcal{E}$, $\mathcal{F}$ are vector bundles, we have now defined vector bundles $\mathcal{E} \oplus \mathcal{F} := \underline{T}_\oplus(\mathcal{E}, \mathcal{F})$, $\mathcal{E} \otimes \mathcal{F} := \underline{T}_\otimes(\mathcal{E}, \mathcal{F})$, and so on for $\mathcal{E}^{\otimes k}$, and $\wedge^k \mathcal{E}$.

Remark: A very similar argument leads to the construction of a vector bundle $\underline{\hom}(\mathcal{E}, \mathcal{F})$, whose fiber at a point $p \in M$ is $\hom_{\mathbb{R}}(\mathcal{E}_p, \mathcal{F}_p)$. (a special case of which gives a construction of $\mathcal{E}^* := \underline{\hom}(\mathcal{E}, \underline{\mathbb{R}})$). However, the argument you'll give in this part doesn't verbatim apply because the functor sending $(V, W) \to \hom_{\mathbb{R}}(V, W)$ is *contravariant* in in the first entry V, (meaning isomorphisms $V_1 \to V_2$ induce, via pre-composition, maps in the *opposite* direction $\hom_{\mathbb{R}}(V_2, W) \to \hom_{\mathbb{R}}(V_1, W)$).

Hence $T_{\text{hom}} := \hom_{\mathbb{R}}(-,-)$ should be thought of as a functor $\mathcal{V}^{op} \times \mathcal{V} \to \mathcal{V}$. Fortunately, there is still a (basically identical) notion for smoothness of functors $T : \underbrace{\mathcal{V}^{op} \times \cdots \times \mathcal{V}^{op}}_{s \text{ times}} \times \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{t \text{ times}} \to \mathcal{V}$. The arguments of this section directly adapt over to give a construction of a vector bundle $\underline{T}(\mathcal{E}_1, \ldots, \mathcal{E}_s; \mathcal{F}_1, \ldots, \mathcal{F}_t)$ for such T.

4. (a) Let $E = [0,1] \times \mathbb{R}/(0,t) \sim (1,-t)$ be the Möbius line bundle defined in class, with $\pi : E \to S^1 = [0,1]/0 \sim 1$ sending $(x,t) \mapsto x$. Verify that E is indeed a line bundle, and prove that E is not isomorphic to the trivial line bundle.

   (b) Let $L = \{(x,v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} | v \in x\}$,[5] $\pi : L \to \mathbb{R}P^n$, $(x,v) \mapsto x$ denote the line bunde introduced in class; we call this bundle the *tautological line bundle on* $\mathbb{R}P^n$. Verify that L is indeed a line bundle.

5. **Linear algebra of tensor products**

   (a) Write in detail the construction of the canonical map $V^* \otimes W \xrightarrow{\alpha} \hom(V,W)$, and give a careful proof that it is an isomorphism if V and W are finite dimensional (in class, we constructed the map and sketchily argued why it was surjective).

   (b) Let $ev : V^* \otimes V \to \mathbb{R}$ be the linear map induced by the bilinear map $\bar{ev} : V^* \times V \to \mathbb{R}$, $(\phi, v) \mapsto \phi(v)$. Given a linear operator $T \in \hom(V,V)$ on a finite-dimensional vector space, define the *trace* of T as $tr(T) := ev(\alpha^{-1}(T))$, where α is the map defined in the previous section.

   Show that this definition agrees with the usual notion of trace, that is if $\underline{v} := \{v_1, \ldots, v_k\}$ is any basis of V and A is the matrix of T with respect to $\underline{v}$, then $tr(T) = tr(A) = \sum_i a_{ii}$.

6. **Exterior algebra 1.**

   (a) *A formula for the determinant of* 3×3 *matrices.* Recall from class that the determinant $\det(T)$ of $T \in \hom_{\mathbb{R}}(V,V)$ is defined as the scalar in $\mathbb{R}$ such that

$$T(v_1) \wedge \cdots \wedge T(v_n) = \det(T) \cdot v_1 \wedge \cdots \wedge v_n,$$

   where $v_1, \ldots, v_n$ is any basis for V.

   Suppose that $\dim V = 3$, and $\underline{v} = (v_1, v_2, v_3)$ is a basis for V. Let $T : V \to V$ be the linear operator defined by

$$\begin{aligned} T(v_1) &= av_1 + dv_2 + gv_3 \\ T(v_2) &= bv_1 + ev_2 + hv_3 \\ T(v_3) &= cv_1 + fv_2 + iv_3. \end{aligned}$$

[5] By $v \in x$, we mean that, if $x = [w]$, then $v \in Span(w)$.

In other words, suppose the matrix of T with respect to $\underline{v}$ is

$$\mathcal{M}(T, \underline{v}) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Derive, using the definition we gave in class with exterior products, a formula for $\det(T)$ in terms of a, b, c, d, e, f, h, and i. (Remark: this formula should have six terms. It is easy to find the formula online, but you should be able to derive it in terms of the definition of determinants given in lecture).

7. **Exterior algebra 2.** For the below problems, let V be a finite-dimensional vector space over $\mathbb{R}$.
   (a) Let $A^k(V) := \text{AltMultiLin}_{\mathbb{R}}(\underbrace{V \times \cdots \times V}_{k \text{ times}}, \mathbb{R})$ be the vector space of alternating multilinear maps from k copies of V to $\mathbb{R}$ (what is its vector space structure). Also let $L^k(V) := \text{MultiLin}_{\mathbb{R}}(\underbrace{V \times \cdots \times V}_{k \text{ times}}, \mathbb{R})$ is the vector space of multilinear maps.

   Prove that there are canonical isomorphisms $A^k(V) \cong \wedge^k V^* \cong (\wedge^k V)^*$. Similarly, prove that there is a canonical isomorphism $L^k(V) \cong (V^*)^{\otimes k} \cong (V^{\otimes k})^*$, and that under these inclusions, the natural map $A^k(V) \hookrightarrow L^k(V)$ is sent to the (dual of) the projection map $(V)^{\otimes k} \to \wedge^k V$. (here we are implicitly using the fact that $(V^{\otimes k})^* \cong (V^*)^{\otimes k}$ and similarly for the wedge product).

   (b) An element $\omega \in A^2(V) = \wedge^2 V^*$ is called *non-degenerate*, or a *linear symplectic form*, if $\omega(v, -) \neq 0 \in \hom_{\mathbb{R}}(V, \mathbb{R})$ for any non-zero $v \in V$. If V is finite-dimensional and V admits a linear symplectic form, prove that $n = \dim V$ is necessarily even, say $n = 2m$.

   (c) Prove that $\omega \in \wedge^2 V^*$ is non-degenerate if and only if $\omega^m \neq 0$ in $\wedge^n V^*$ (where $n = 2m$).

8. Give a careful construction of the exterior differentiation operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ using local coordinates; show that this definition is independent of local coordinates and is well-defined.

9. Let M be a manifold. Prove that d satisfies the formula $d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^k \alpha \wedge d(\beta)$, where $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$.

10. Prove that d commutes with pullback; that is, $d \circ f^* = f^* \circ d$ for any smooth $f : M \to N$.