## Homework 5

EXERCISE 5.1. Let $(E, \pi_E)$ and $(F, \pi_F)$ be (smooth) vector bundles over a common base $M$. A *vector bundle morphism over* $M$ from $E$ to $F$ is a $C^\infty$ map $f\colon E \longrightarrow F$, compatible with projection, meaning that $\pi_F \circ f = \pi_E$, so that on each fiber $f_p\colon E_p \longrightarrow F_p$ is a linear map of vector spaces. An *isomorphism of vector bundles* $E \cong F$ over potentially different manifolds $M$ and $N$ is a diffeomorphism $\overline{f}\colon M \longrightarrow N$ and a vector bundle morphism covering $\overline{f}$ which is an isomorphism on each fiber. We say $E$ and $F$ are *isomorphic over* $M$ if the isomorphism of vector bundles covers the identity map; e. g., if the vector bundle morphism $f\colon E \longrightarrow F$ satisfies $\pi_F \circ f = \pi_E$, so $f$ maps $E_p$ to $F_p$.

Let $(E, \pi\colon E \longrightarrow M)$ be a rank $k$ vector bundle. We say a collection of sections $s_1, \dots, s_k \in \Gamma(E)$ is *linearly independent* if $s_1(p), \dots, s_k(p)$ are linearly independent in $E_p$ for each $p \in M$ (in particular, no $s_i(p)$ should be zero). Prove that there is an isomorphism of vector bundles $E \cong \underline{\mathbb{R}}^k$ over $M$ if and only if $E$ has a basis of linearly independent sections $s_1, \dots, s_k$.

*Solution.* First, suppose that $f\colon \underline{\mathbb{R}}^k \longrightarrow E$ is a vector bundle isomorphism over $M$. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^k$ and observe that $s_i(m) = (m, e_i)$ defines a basis of sections for $\underline{\mathbb{R}}^k$. Then $\{f \circ s_i\}$ is a set of sections of $E$ and because $f$ is fiberwise an isomorphism these sections form a basis for $E$.

Conversely, suppose that $s_1, \dots, s_k$ is a basis of sections for $E$. Define a map $f\colon \underline{\mathbb{R}}^k \longrightarrow E$ by setting $f(m, (x^1, \dots, x^k)) = \sum_i x^i s_i(m)$. This map clearly preserves fibers and on each fiber it is an isomorphism because $s_1, \dots, s_k$ was assumed to be a basis for $E$.

EXERCISE 5.2. Let $M$ be a smooth manifold and $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ an open cover of $M$. A set of *transition* (or *gluing*) *data* (of rank $k$) for the cover $\mathscr{U}$ is a collection of $C^\infty$ maps

$$\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_k(\mathbb{R}), \quad \text{for all } \alpha, \beta \in I,$$

satisfying

- For any $\alpha, \beta \in I$ and $x \in U_\alpha \cap U_\beta$ we have $\Phi_{\alpha\beta}(x) \cdot \Phi_{\beta\alpha}(x) = I$.
- For any $\alpha, \beta, \gamma \in I$ and $x \in U_\alpha \cap U_\beta \cap U_\gamma$ we have $\Phi_{\gamma\alpha}(x) = \Phi_{\gamma\beta}(x) \cdot \Phi_{\beta\alpha}(x)$.

(i) Show, as asserted in class, that, given a collection of transition data $\mathscr{T} := \{\Phi_{\alpha\beta}\}_{\alpha,\beta \in I}$ for the cover $\mathscr{U}$, there is a $C^\infty$ rank $k$ vector bundle $\mathscr{E}_{\mathscr{U},\mathscr{T}}$ over $M$, formed by gluing together trivial vector bundles $U_\alpha \times \mathbb{R}^k$ via the transition maps:

$$\mathscr{E}_{\mathscr{U},\mathscr{T}} := \coprod_{U_\alpha \in \mathscr{U}} (U_\alpha \times \mathbb{R}^k)/\{(U_\alpha, p, v) \sim (U_\beta, p, \Phi_{\alpha\beta}(p)(v))\}$$

with projection to $M \cong \coprod_{U_\alpha \in \mathscr{U}} U_\alpha/\{(U_\alpha, p) \sim (U_\beta, p)\}$ given by $\pi\colon [(U_\alpha, p, v)] \longmapsto [(U_\alpha, p)]$.

(ii) Show that any vector bundle $E$ on $M$ is isomorphic as vector bundles to $\mathscr{E}_{\mathscr{U},\mathscr{T}}$ for some cover $\mathscr{U}$ and some transition data $\mathscr{T}$, in a manner covering the canonical map $M_{\mathscr{U}} := \coprod U_\alpha/\!\sim \longrightarrow M$.

(iii) Let $\mathscr{U} = \{U_i\}_{i \in I}$ be a cover and $\mathscr{T} = \{\Phi_{\alpha\beta}\}$ a set of transition data of rank $k$. A *(Čech)* 0*–cochain* on $\mathscr{U}$ is a collection of function $\mathscr{F} = \{f_\alpha\colon U_\alpha \longrightarrow \mathrm{GL}_k(\mathbb{R})\}$. Given such a Čech 0–cochain, define $\mathscr{F}\mathscr{T} := \{f_\alpha \circ \Phi_{\alpha\beta} \circ f_\beta^{-1}\}_{\alpha,\beta \in I}$. Show that $\mathscr{E}_{\mathscr{U},\mathscr{T}}$ and $\mathscr{E}_{\mathscr{U},\mathscr{F}\mathscr{T}}$ are isomorphic as vector bundles.

(iv) Giving a cover $\mathscr{U}$ and transition data $\mathscr{T}$, a *refinement* of the cover $\mathscr{U}$ is a new cover $\mathscr{U}' = \{V_\kappa\}_{\kappa \in \mathcal{J}}$ along with a choice of $\alpha \in I$ for each $\kappa \in \mathcal{J}$ such that $V_\kappa \subset U_\alpha$. Given a refinement of $\mathscr{U}$, call it $\mathscr{U}'$, the *induced transition data on the refinement* is

$$\mathscr{T}|_{\mathscr{U}'} = \{\Phi_{\alpha\beta} : \alpha, \beta \in I \text{ are the chosen indices such that } V_\kappa \subset U_\alpha,\ V_\tau \subset U_\beta\}_{\kappa,\tau \in \mathcal{J}}.$$

Show that there is an isomorphism $\mathscr{E}_{\mathscr{U}',\mathscr{T}|_{\mathscr{U}'}} \cong \mathscr{E}_{\mathscr{U},\mathscr{T}}$ of vector bundles.

(v) Let $\mathrm{Vect}_k(M)$ denote the set of smooth rank $k$ vector bundles on $M$ up to isomorphism. Show that $\mathrm{Vect}^k(M) \cong \{(\mathscr{U}, \mathscr{T})\}/\!\sim$ where $(\mathscr{U}_1, \mathscr{T}_1) \sim (\mathscr{U}_2, \mathscr{T}_2)$ if, after passing to a common refinement $\mathscr{U}'$ of $\mathscr{U}_1$ and $\mathscr{U}_2$, there is a Čech 0–cochain $\mathscr{F}$ on $\mathscr{U}'$ such that

$$\mathscr{T}_1|_{\mathscr{U}'} = \mathscr{F}(\mathscr{T}_2|_{\mathscr{U}'}).$$

*Solution.* First, write $M' = \coprod_{U_\alpha \in \mathscr{U}} U_\alpha / \{(U_\alpha, p) \sim (U_\beta, p)\}$. This is clearly covered by the images of the $U_\alpha$ and the map $f\colon M \longrightarrow M'$ with $f(p) = [(U_\alpha, p)]$ if $p \in U_\alpha$ is well defined and induces homeomorphisms $f|_{U_\alpha}\colon U_\alpha \longrightarrow f(U_\alpha)$. Furthermore $f$ is bijective and therefore a homeomorpism $M \cong M'$. Similarly, the images of the $U_\alpha \times \mathbb{R}^k$ cover $\mathscr{E}_{\mathscr{U},\mathscr{T}}$ and are homeomorphic to $U_\alpha \times \mathbb{R}^k$.

The inverse image $\pi^{-1}(f(U_\alpha))$ is the image of $U_\alpha \times \mathbb{R}^k$ in $\mathscr{E}_{\mathscr{U},\mathscr{T}}$. This gives local trivializations of $\mathscr{E}_{\mathscr{U},\mathscr{T}}$ and we first check that the the cover of $\mathscr{E}_{\mathscr{U},\mathscr{T}}$ by the images of the $U_\alpha \times \mathbb{R}^k$ together with their homeomorphisms to $U_\alpha \times \mathbb{R}^k$ is a smooth atlas for $\mathscr{E}_{\mathscr{U},\mathscr{T}}$. To this end consider the transition maps

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^k,\ (p, v) \longmapsto (p, \Phi_{\alpha\beta}(p)(v)).$$

These are smooth since $\Phi_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_k(\mathbb{R})$ is smooth by assumption and $\Phi_{\alpha\beta}(p)$ is linear and therefore smooth, too. This identification of the transition maps also proves that $\mathscr{E}_{\mathscr{U},\mathscr{T}}$ since the transition maps are fiberwise linear.

Now, let $\pi\colon E \longrightarrow M$ be a smooth vector bundle over $M$ and let $\{U_\alpha\}_\alpha$ be an atlas for $M$ which trivializes $E$. That is, we have commutative diagrams

$$\begin{array}{ccc} E|_{U_\alpha} & \xrightarrow{\ \phi_\alpha\ } & U_\alpha \times \mathbb{R}^k \\ & {\scriptstyle\pi}\searrow & \downarrow \\ & & U_\alpha \end{array}$$

where each $\phi_\alpha$ is a fiberwise linear diffeomorphism. For indices $\alpha$ and $\beta$ we have commutative diagrams

$$\begin{array}{ccccc} (U_\alpha \cap U_\beta) \times \mathbb{R}^k & \xrightarrow{\ \phi_\alpha^{-1}\ } & E|_{U_\alpha \cap U_\beta} & \xrightarrow{\ \phi_\beta\ } & (U_\alpha \cap U_\beta) \times \mathbb{R}^k \\ & \searrow & \downarrow & \swarrow & \\ & & U_\alpha \cap U_\beta & & \end{array}$$

and we define $\widetilde{\Phi}_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$. This is a smooth fiberwise linear map over $U_\alpha \cap U_\beta$, so we have a smooth map

$$\Phi_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_k(\mathbb{R}),\ p \longmapsto \phi_\beta|_{\{p\}\times\mathbb{R}^k} \circ \phi_\alpha^{-1}|_{\{p\}\times\mathbb{R}^k}.$$

In this way we obtain transition data on the cover $\mathscr{U} = \{U_\alpha\}$ of $M$; the cocycle conditions for $\mathscr{T} = \{\Phi_{\alpha\beta}\}$ follow immediately from its definition. It remains to show that $\mathscr{E}_{\mathscr{U},\mathscr{T}}$ is isomorphic to $E$. But because of the definition of $\mathscr{E}_{\mathscr{U},\mathscr{T}}$ the maps $\phi_\alpha^{-1}\colon U_\alpha \times \mathbb{R}^k \longrightarrow E|_{U_\alpha} \subset E$ assemble into a smooth map $\mathscr{E}_{\mathscr{U},\mathscr{T}} \longrightarrow E$ which covers the map $M_{\mathscr{U}} \longrightarrow M$. This map is a local diffeomorphism and fiberwise linear since it restricts $\phi_\alpha$ on $E|_{U_\alpha}$. It is also bijective directly from the definition of the equivalence relation defining $\mathscr{E}_{\mathscr{U},\mathscr{T}}$. It follows that it is a vector bundle isomorphism $\mathscr{E}_{\mathscr{U},\mathscr{T}} \cong E$.

Let $\mathscr{F} = \{f_\alpha\}_\alpha$ be a Čech 0–cochain on $\mathscr{U} = \{U_\alpha\}_\alpha$ and assume we have transition data $\mathscr{T} = \{\Phi_{\alpha\beta}\}_{\alpha\beta}$ on $\mathscr{U}$. Define a diffeomorphism $f\colon \coprod U_\alpha \times \mathbb{R}^k \longrightarrow \coprod U_\alpha \times \mathbb{R}^k$ by

$$(U_\alpha, p, v) \longmapsto (U_\alpha, p, f_\alpha(p)(v))$$

and observe that it is linear for fixed $U_\alpha$ and $p$. Hence, it is enough to show that this map descends to a bijective map $\mathscr{E}_{\mathscr{U},\mathscr{T}} \longrightarrow \mathscr{E}_{\mathscr{U},\mathscr{F}\mathscr{T}}$. If $p \in U_\alpha \cap U_\beta$ then

$$[U_\alpha, p, (f_\alpha(p) \circ \Phi_{\alpha\beta}(p))(v)] = [U_\alpha, p, (f_\alpha(p) \circ \Phi_{\alpha\beta}(p) \circ f_\beta(p)^{-1})(f_\beta(p)(v))] \sim [U_\beta, p, f_\beta(p)(v)]$$

in $\mathscr{E}_{\mathscr{U},\mathscr{F}\mathscr{T}}$. Hence, $f$ descends to a map $\mathscr{E}_{\mathscr{U},\mathscr{T}} \longrightarrow \mathscr{E}_{\mathscr{U},\mathscr{F}\mathscr{T}}$ as does its inverse. Therefore, we have an isomorphism $\mathscr{E}_{\mathscr{U},\mathscr{T}} \cong \mathscr{E}_{\mathscr{U},\mathscr{F}\mathscr{T}}$ of vector bundles.

Assume given a cover $\mathscr{U} = \{U_\alpha\}$ with transition data $\mathscr{T}$, a cover $\mathscr{U}' = \{V_\kappa\}_\kappa$ and function $\kappa \longrightarrow f(\kappa)$ such that $V_\kappa \subset U_{f(\kappa)}$ witnessing that $\mathscr{U}'$ is a refinement of $\mathscr{U}$. Define $\varphi\colon \coprod V_\kappa \times \mathbb{R}^k \longrightarrow \coprod U_\alpha \times \mathbb{R}^k$ by

$$(V_\kappa, p, v) \longmapsto (U_{f(\kappa)}, p, v).$$

This defines a local diffeomorphism so we are reduced to checking that it descends to a bijective map $\mathscr{E}_{\mathscr{U}',\mathscr{T}|_{\mathscr{U}'}} \longrightarrow \mathscr{E}_{\mathscr{U},\mathscr{T}}$. For this, observe that for $p \in V_\kappa \cap V_{\kappa'} \subset U_{f(\kappa)} \cap U_{f(\kappa')}$ we have

$$[\varphi(V_\kappa, p, v)] = [U_{f(\kappa)}, p, v] = [U_{f(\kappa')}, p, \Phi_{f(\kappa)f(\kappa')}(p)(v)] = [\varphi(V_{\kappa'}, p, \Phi_{f(\kappa)f(\kappa')}(v))].$$

Consequently, $\varphi$ descends to a local diffeomorphism $\widetilde{\varphi}\colon \mathscr{E}_{\mathscr{U}',\mathscr{T}|_{\mathscr{U}'}} \longrightarrow \mathscr{E}_{\mathscr{U},\mathscr{T}}$. Now, if $p \in V_\kappa$ and $q \in V_{\kappa'}$ are such that $[U_{f(\kappa)}, p, v] = [U_{f(\kappa')}, q, w]$, then $p = q \in U_{f(\kappa)\cap f(\kappa')} \supset V_\kappa \cap V_{\kappa'}$ and $w = \Phi_{f(\kappa)f(\kappa')}(p)(v)$. But this means that $[V_\kappa, p, v] = [V_{\kappa'}, q, w]$ and therefore $\widetilde{\varphi}$ is injective. Additionally, for $[U_\alpha, p, v] \in \mathscr{E}_{\mathscr{U},\mathscr{T}}$ there is some $\kappa$ such that $p \in V_\kappa$. Then we find

$$\widetilde{\varphi}([V_\kappa, p, \Phi_{\alpha f(\kappa)}(p)(v)]) = [U_{f(\kappa)}, p, \Phi_{\alpha f(\kappa)}(p)(v)] = [U_\alpha, p, v]$$

and we conclude that $\widetilde{\varphi}$ is surjective as well.

Lastly, parts (iii) and (iv) show that we have a well defined map $\{(\mathscr{U}, \mathscr{T})\}/\sim \longrightarrow \mathrm{Vect}_k(M)$ which part (ii) shows to be surjective. If $f\colon \mathscr{E}_{\mathscr{U}_1,\mathscr{T}_1} \longrightarrow \mathscr{E}_{\mathscr{U}_2,\mathscr{T}_2}$ is an isomorphism of vector bundles, let $\mathscr{U}_1 = \{U_\alpha\}_\alpha$ and $\mathscr{U}_2 = \{V_\beta\}_\beta$. Then $\mathscr{U}' = \{U_{\alpha\beta} = U_\alpha \cap V_\beta\}_{\alpha\beta}$ is a common refinement of $\mathscr{U}_1$ and $\mathscr{U}_2$. We have a chain of fiberwise linear diffeomorphisms

$$U_{\alpha\beta} \times \mathbb{R}^k \cong \mathscr{E}_{\mathscr{U}_1,\mathscr{T}_1}|_{U_{\alpha\beta}} \xrightarrow{f} \mathscr{E}_{\mathscr{U}_2,\mathscr{T}_2}|_{U_{\alpha\beta}} \cong U_{\alpha\beta} \times \mathbb{R}^k$$

and therefore a Čech 0–cochain $\mathscr{F}$ on $\mathscr{U}'$. It is immediate from the definition of $\mathscr{F}$ that $\mathscr{T}_1|_{\mathscr{U}'} = \mathscr{F}(\mathscr{T}_2|_{\mathscr{U}'})$.

Exercise 5.3. In class, we have indicated or sketched how many vector space operations, such as direct sum, tensor product, self tensor/exterior product, quotient, etc. induce operations on vector bundles. In this homework assignment we will give a streamlined proof of all such constructions, using some category theory.

First, if $\mathscr{C}_1, \ldots, \mathscr{C}_k$ are categories, the *product category* $\mathscr{C}_1 \times \cdots \times \mathscr{C}_k$ has as objects elements of the cartesian product $\mathrm{ob}(\mathscr{C}_1) \times \cdots \times \mathrm{ob}(\mathscr{C}_k)$, and has as morphisms, for a pair of tuples $(X_1, \ldots, X_k)$ and $(Y_1, \ldots, Y_k)$, the set

$$\mathrm{Hom}_{\mathscr{C}_1\times\cdots\times\mathscr{C}_k}((X_1, \ldots, X_k), (Y_1, \ldots, Y_k)) = \mathrm{Hom}_{\mathscr{C}_1}(X_1, Y_1) \times \cdots \times \mathrm{Hom}_{\mathscr{C}_k}(X_k, Y_k),$$

with composition given component-wise. We use the shorthand $\mathscr{C}^k$ for the product of $k$ copies of $\mathscr{C}$.

A *$k$–ary functor* from $\mathscr{C}$ to $\mathscr{D}$ is a functor $F\colon \mathscr{C}^k \longrightarrow \mathscr{D}$. Now, let $\mathscr{V} = \mathrm{Vect}^{\mathrm{iso}}(\mathbb{R})$ be the category whose objects are finite dimensional $\mathbb{R}$–vector spaces and whose morphisms are the $\mathbb{R}$–linear isomorphisms, with composition given by composition of linear maps. Note that in this category $\mathrm{Hom}_{\mathscr{V}}(V, V) = \mathrm{GL}(V)$.

(i) Check that the product category $\mathscr{C}_1 \times \cdots \times \mathscr{C}_k$ is indeed a category if $\mathscr{C}_1, \ldots, \mathscr{C}_k$ are.

(ii) Show that in $\mathscr{V}$, the set $\mathrm{Hom}_{\mathscr{V}}(V, W)$ has the structure of a $C^\infty$ manifold for any pair of finite dimensional vector spaces $V$ and $W$. Moreover, check that the composition map $\mathrm{Hom}_{\mathscr{V}}(W, Z) \times \mathrm{Hom}_{\mathscr{V}}(V, W) \longrightarrow \mathrm{Hom}_{\mathscr{V}}(V, Z)$ is a $C^\infty$ map of manifolds.

(iii) We say a $k$–ary functor $T\colon \mathscr{V}^k \longrightarrow \mathscr{V}$ is *smooth* if, for any pair of objects $(V_1, \ldots, v_k)$ and $(W_1, \ldots, W_k)$ in $\mathscr{V}^k$, the induced map on morphism spaces

$$\mathrm{Hom}_{\mathscr{V}}(V_1, W_1) \times \cdots \times \mathrm{Hom}_{\mathscr{V}}(V_k, W_k) \longrightarrow \mathrm{Hom}_{\mathscr{V}}(T(V_1, \ldots, V_k), T(W_1, \ldots, W_k))$$

is a $C^\infty$ map of manifolds.

Show that the following give smooth $k$–ary functors on $\mathscr{V}$:

- $(V, W) \longmapsto V \oplus W$,
- $(V, W) \longmapsto V \otimes W$,
- $(V, W) \longmapsto \mathrm{Hom}(V, W)$,
- $V \longmapsto V^{\otimes n}$ for any $n \geq 0$, and
- $V \longmapsto \bigwedge^n V$ for any $n \geq 0$.

(iv) Given a tuple of vector bundles $(\mathscr{E}_1, \ldots, \mathscr{E}_k)$ over manifold $M$, and a smooth $k$–ary functor $T$ as above, directly construct a smooth vector bundle $\underline{T}(\mathscr{E}_1, \ldots, \mathscr{E}_k)$ with $\underline{T}(\mathscr{E}_1, \ldots, \mathscr{E}_k)_p = T(\mathscr{E}_{1,p}, \ldots, \mathscr{E}_{2,p})$.

(v) Give a construction of the vector bundle $\underline{T}(\mathscr{E}_1, \ldots, \mathscr{E}_k)$ in terms of transition data, using as input the transition data associated to $\mathscr{E}_1, \ldots, \mathscr{E}_k$ for some cover, and explain why the result is isomorphic to the construction above.

*Solution.* Since composition of morphisms in $\mathscr{C}_1 \times \cdots \times \mathscr{C}_k$ is defined componentwise it is associative with identity morphism $(1_{\mathscr{C}_1}, \ldots, 1_{\mathscr{C}_k})$. This is enough to show that $\mathscr{C}_1 \times \cdots \times \mathscr{C}_k$ is a category.

The set $\mathrm{Hom}_{\mathsf{Vect}}(V, W)$ of arbitrary homomorphisms between two vector spaces $V$ and $W$ is a vector space itself. On a previous homework assignment we saw that any finite dimensional vector space has a smooth structure given by any isomorphism to $\mathbb{R}^n$ and this smooth structure is independent of the choice of isomorphism. Since $\mathrm{Hom}_{\mathscr{V}}(V, W)$ is an open subset of $\mathrm{Hom}_{\mathsf{Vect}}(V, W)$ it inherits a canonical smooth structure this way. To check that composition of $\mathscr{V}$ is smooth it is then enough to show that composition $\mathrm{Hom}_{\mathsf{Vect}}(W, Z) \times \mathrm{Hom}_{\mathsf{Vect}}(V, W) \longrightarrow \mathrm{Hom}_{\mathsf{Vect}}(V, Z)$ of linear maps is smooth. But choosing bases for $V$, $W$ and $Z$ this composition is just given by matrix multiplication which is smooth.

To check that the given functors are smooth, let $V_1$, $V_2$ and $W_1$, $W_2$ be vector spaces with chosen bases $\mathscr{B}(V_i)$ and $\mathscr{V}(W_i)$ respectively.

- A basis of $V_i \oplus W_i$ could be $\{(v, w) : v \in \mathscr{B}(V_i), w \in \mathscr{B}(W_i)\}$. Given homomorphisms $f\colon V_1 \longrightarrow V_2$ and $g\colon W_1 \longrightarrow W_2$ represented by matrices $M_f$ and $M_g$ respectively, the induced homomorphism $f \oplus g\colon V_1 \oplus W_1 \longrightarrow V_2 \oplus W_2$ is represented by the block diagonal matrix

$$M_{f \oplus g} = \left(\begin{array}{c|c} M_f & 0 \\ \hline 0 & M_g \end{array}\right)$$

  But the assignment $(M_f, M_g) \longmapsto M_{f \oplus g}$ defines a smooth map $\mathbb{R}^{n_2 \times n_1} \times \mathbb{R}^{m_2 \times m_1} \longrightarrow \mathbb{R}^{(n_2+m_2) \times (n_1+m_1)}$ where $n_i = \dim(V_i)$ and $m_i = \dim(W_i)$. This restricts to a smooth map

$$\mathrm{Hom}_{\mathscr{V}}(V_1, W_1) \times \mathrm{Hom}_{\mathscr{V}}(V_2, W_2) \longrightarrow \mathrm{Hom}_{\mathscr{V}}(V_1 \oplus W_1, V_2 \oplus W_2).$$

  We conclude that $\_ \oplus \_$ is a smooth bifunctor.
- Given morphisms $f\colon V_1 \longrightarrow V_2$ and $g\colon W_1 \longrightarrow W_2$ the induced morphism $V_1 \otimes W_1 \longrightarrow V_2 \otimes W_2$ is given by $v \otimes w \longmapsto f(v) \otimes g(w)$. To check that the map $(f, g) \longmapsto f \otimes g$ is smooth observe that the matrix representation of $f \otimes g$ is given by the Kronecker product of the matrices $M_f$ and $M_g$. In particular, its coefficients are bilinear functions in the coefficients of $M_f$ and $M_g$. This implies that $\_ \otimes \_$ is a smooth bifunctor.
- For given linear maps $f\colon V_2 \longrightarrow V_1$ and $g\colon W_1 \longrightarrow W_2$ the induced map on morphism spaces $\mathrm{Hom}(V_1, W_1) \longrightarrow \mathrm{Hom}(V_2, W_2)$ is given by

$$\phi \longmapsto g_* f^*(\phi) = g \circ \phi \circ f.$$

  This mapping $(f, g) \longmapsto g_* f^*$ is again bilinear in $f$ and $g$ and therefore smooth.
- The action of the functor $V \longmapsto V^{\otimes n}$ on morphisms $f\colon V \longrightarrow W$ is given by $f \longmapsto f^{\otimes n}$. In terms of matrices the linear map $f^{\otimes n}$ is given by taking the Kronecker product of the matrix of $f$ with itself $n$ times. In particular, the entries of the matrix of $f^{\otimes n}$ depend polynomially on the entries of the matrix of $f$. It follows that $(\_)^{\otimes n}$ defines a smooth functor as well.

- Again, the action of the functor $V \longmapsto V^{\wedge n}$ on morphisms $f\colon V \longrightarrow W$ is given by $f \longmapsto f^{\wedge n}$. The matrix of $f^{\wedge n}$ can be computed as the matrix of the determinants of the minors of $M_f$ with respect to the standard bases of the exterior products. In particular, its entries depend polynomially on $M_f$ and therefore $(\_)^{\wedge n}$ defines a smooth functor as well.

Let $\mathscr{E}_1, \ldots, \mathscr{E}_k$ be vector bundles over $M$ of ranks $n_1, \ldots, n_k$ respectively and let $T$ be a smooth $k$–ary functor. Choose an atlas $\{U_\alpha\}_\alpha$ for $M$ trivializing the $\mathscr{E}_i$ simultaneously. That is, we have fiberwise linear diffeomorphisms $\varphi_{i\alpha}\colon \mathscr{E}_i|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^{n_i}$ where we suppress diffeomorphisms $U_\alpha \cong \mathbb{R}^{\dim(M)}$ from the notation. Define

$$\underline{T}(\mathscr{E}_1, \ldots, \mathscr{E}_k) = \mathscr{E} = \coprod_{x \in M} T(\mathscr{E}_{1,x}, \ldots, \mathscr{E}_{k,x})$$

together with the canonical map $\pi\colon \mathscr{E} \longrightarrow M$. Define a topology on $\mathscr{E}$ by declaring the maps

$$\psi_\alpha\colon \pi^{-1}(U_\alpha) = \coprod_{x \in U_\alpha} T(\mathscr{E}_{1,x}, \ldots, \mathscr{E}_{k,x}) \longrightarrow U_\alpha \times T(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k})$$

with $\psi_\alpha(x, \_) = T(\varphi_{1\alpha,x}, \ldots, \varphi_{k\alpha,x})$ to be homeomorphisms. The transition maps

$$(U_\alpha \cap U_\beta) \times T(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}) \longrightarrow (U_\alpha \cap U_\beta) \times T(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k})$$

are then given by

$$(\psi_\beta \psi_\alpha^{-1})(x, v) = (T(\varphi_{1\beta,x}, \ldots, \varphi_{k\beta,x}) T(\varphi_{1\alpha,x}, \ldots, \varphi_{k\alpha,x})^{-1})(v) = T(\varphi_{1\beta,x}\varphi_{1\alpha,x}^{-1}, \ldots, \varphi_{k\beta,x}\varphi_{k\alpha,x}^{-1})(v)$$

because $T$ is an $k$–ary functor on $\mathscr{V}$. In particular, the transition maps are smooth and fiberwise linear and we conclude as usual that $\mathscr{E}$ is a smooth manifold. In fact, we directly constructed a vector bundle atlas for $\mathscr{E}$.

Let $\{U_\alpha\}_\alpha$ be the same atlas for $M$ as in the previous part and let $\{\Phi^{(i)}_{\alpha\beta}\}_{\alpha\beta}$ be the transition data for $\mathscr{E}_i$ on $\{U_\alpha\}$. Define

$$\Phi_{\alpha\beta} = T(\Phi^{(1)}_{\alpha\beta}, \ldots, \Phi^{(k)}_{\alpha\beta}).$$

This family of functions satisfies the cocycle conditions because $\mathscr{T}$ is a $k$–ary functor $\mathscr{V}^k \longrightarrow \mathscr{V}$. Hence, we have transition data $\{\Phi_{\alpha\beta}\}_{\alpha\beta}$ and therefore a vector bundle $\mathscr{E}'$. In fact, remembering that $\Phi^{(i)}_{\alpha\beta} = \varphi_{i\beta}\varphi_{i\alpha}^{-1}$, the transition data for $\mathscr{E}'$ are exactly the same as the transition data for the bundle $\mathscr{E}$ constructed in part (iv). By the previous exercise we conclude that $\mathscr{E}' \cong \mathscr{E}$.

Exercise 5.4. Let $E = [0,1] \times \mathbb{R}/(0,t) \sim (1,-t)$ be the Möbius line bundle defined in class, with the projection map $\pi\colon E \longrightarrow S^1 = [0,1]/0 \sim 1$ sending $(x,t) \longmapsto x$. Verify that $E$ is indeed a line bundle, and prove that $E$ is not isomorphic to the trivial line bundle.

Let $L = \{(x,v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in x\}$ with the projection $\pi\colon L \longrightarrow \mathbb{RP}^n$ sending $(x,v) \longmapsto x$ denote the line bundle introduced in class; we call this bundle the *tautological line bundle on* $\mathbb{RP}^n$. Verify that $L$ is indeed a line bundle.

*Solution.* Let $U_1 = (0,1) \subset S^1$ and $U_2 = ([0,1/2) \cup (1/2,1])/0 \sim 1 \subset S^1$ be the standard cover of $S^1$. Define trivializations for $E$ by

$$E|_{U_1} \longrightarrow U_1 \times \mathbb{R},\ [x,t] \longmapsto (x,t)$$
$$E|_{U_2} \longrightarrow U_2 \times \mathbb{R},\ [x,t] \longmapsto \begin{cases} (x,t) & x \in [0,1/2) \\ (x,-t) & x \in (1/2,1] \end{cases}$$

Observe that the trivialization over $U_2$ is well defined and smooth because $[0,t] = [1,-t]$ in $E$. These trivializations are fiberwise linear and so we have a vector bundle structure on $E$. To see that the Möbius strip is not trivial note that $E \smallsetminus s_0(S^1)$, the complement of the zero section in $E$, is connected. The complement of the zero section in the trivial bundle is disconnected, so $E$ cannot be homeomorphic to $S^1 \times \mathbb{R}$ via a homeomorphism preserving the zero section. But any vector bundle isomorphism would need to preserve the zero section.

We use homogeneous coordinates $[x_0 : \cdots : x_n]$ on $\mathbb{RP}^n$. Then $\mathbb{RP}^n$ admits an atlas $\{U_i\}_{i=0,\ldots,n}$ with

$$U_i = \{[x_0 : \cdots : x_n] : x_i = 1\}$$

and $\mathbb{R}^n \cong U_i$ via

$$(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \longmapsto [x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n].$$

We define a trivialization of $L$ on $U_i$ by

$$L|_{U_i} \longrightarrow U_i \times \mathbb{R}, \ ([x], v) \longmapsto \left(x, \frac{\langle x, v\rangle}{\sqrt{\langle x, x\rangle}}\right)$$

where $\langle \_, \_\rangle$ denotes the usual Euclidean scalar product. Note that the expression $\langle x, v\rangle/\sqrt{\langle x, v\rangle}$ is invariant under scaling of $x$. Therefore, this map is a well defined and smooth. The condition $v \in x$, that is, $v = \alpha x$ for some $\alpha \in \mathbb{R}$, in the definition of $L$ ensures that it is fiberwise an isomorphism. In this way we obtain a vector bundle atlas for $L$.

Exercise 5.5. Write in detail the construction of the canonical map $\alpha\colon V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$, and give a careful proof that it is an isomorphism if $V$ and $W$ are finite dimensional.

Let $\mathrm{ev}\colon V^* \otimes V \longrightarrow \mathbb{R}$ be the linear map induced by the bilinear map $(\phi, v) \longmapsto \phi(v)$. Given a linear operator $T \in \mathrm{Hom}(V, V)$ on a finite dimensional vector space, define the *trace* of $T$ as $\mathrm{tr}(T) = \mathrm{ev}(\alpha^{-1}(T))$, where $\alpha$ is the map defined in the previous section. Show that this definition agrees with the usual notion of trace.

*Solution.* Define the map $\alpha\colon V^* \times W \longrightarrow \mathrm{Hom}(V, W)$ by setting $\alpha(f, w)(v) = f(v)w$. This is clearly linear in $w$ and

$$\alpha(f_1 + \lambda f_2, w) = (f_1 + \lambda f_2)(v)w = \alpha(f_1, w) + \lambda\alpha(f_2, w).$$

Hence, $\alpha$ is bilinear and we have a linear map $V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$ with $f \otimes w \longmapsto f(\_)w$. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be bases for $V$ and $W$ respectively. Let $\{v_i^*\}$ be the dual basis for $V^*$. Then the set $\{v \longmapsto v_i^*(v)w_j\}_{ij}$ is a basis for $\mathrm{Hom}(V, W)$ because $V$ and $W$ are finite dimensional and $\{v_i^* \otimes w_j\}_{ij}$ is a basis for $V^* \otimes W$. Compute

$$v_i^* \otimes w_j \longmapsto \alpha(v_i^*, w_j)(v) = v_i^*(v)w_j.$$

That is, $f \otimes w \longmapsto f(\_)w$ is a bijection between the chosen bases of $\mathrm{Hom}(V, W)$ and $V^* \otimes W$ and therefore an isomorphism.

Let $T \in \mathrm{Hom}(V, V)$ and write $T_{ij} = v_i^*(T(v_j))$. We compute

$$\sum_{ij} \alpha(T_{ij} v_j^* \otimes v_i)(v) = \sum_{ij} T_{ij} v_j^*(v) v_i = \sum_j T(v_j) v_j^*(v) = T(v).$$

Therefore, $\alpha(\sum_{ij} T_{ij} v_j^* \otimes v_i) = T$ and

$$\mathrm{ev}(\alpha^{-1}(T)) = \sum_{ij} T_{ij} v_j^*(v_i) = \sum_i T_{ii} = \mathrm{tr}(T).$$

EXERCISE 5.6. Recall from class that the determinant $\det(T)$ of $T \in \mathrm{Hom}(V,V)$ is defined as the scalar in $\mathbb{R}$ such that

$$T(v_1) \wedge \cdots \wedge T(v_n) = \det(T) \cdot v_1 \wedge \cdots \wedge v_n$$

where $v_1, \ldots, v_n$ is any basis for $V$.

Suppose that $\dim(V) = 3$ and $\underline{v} = (v_1, v_2, v_3)$ is a basis for $V$. Let $T\colon V \longrightarrow V$ be the linear operator with matrix

$$\mathscr{M}(T, \underline{v}) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Derive, using the definition we gave in class with exterior products, a formula for $\det(T)$ in terms of $a, \ldots, i$.

*Solution.* We compute

$$\begin{aligned} T(v_1) \wedge T(v_2) \wedge T(v_3) &= (av_1 + dv_2 + gv_3) \wedge (bv_1 + ev_2 + hv_3) \wedge (cv_1 + fv_2 + iv_3) = \\ &= ((ae - bd)v_1 \wedge v_2 + (dh - eg)v_2 \wedge v_3 + (ah - bg)v_1 \wedge v_3) \wedge (cv_1 + fv_2 + iv_3) = \\ &= ((ae - bd)i + (dh - eg)c - (ah - bg)f)v_1 \wedge v_2 \wedge v_3 = \\ &= \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} v_1 \wedge v_2 \wedge v_3. \end{aligned}$$

EXERCISE 5.7. Let $V$ be a finite dimensional vector space over $\mathbb{R}$.

(i) Let $A^k(V)$ be the vector space of alternating multilinear maps $V^k \longrightarrow \mathbb{R}$. Also, let $L^k(V)$ be the vector space of multilinear maps $V^k \longrightarrow \mathbb{R}$. Prove that there are canonical isomorphisms $A^k(V) \cong \bigwedge^k V^* \cong (\bigwedge^k V)^*$. Similarly, prove that there is a canonical isomorphism $L^k(V) \cong (V^*)^{\otimes k} \cong (V^{\otimes k})^*$, and that under these isomorphisms, the natural inclusion $A^k(V) \hookrightarrow L^k(V)$ is sent to the dual of the projection map $V^{\otimes k} \longrightarrow \bigwedge^k V$.

(ii) An element $\omega \in A^2(V)$ is called *non-degenerate*, or a *linear symplectic form*, if $\omega(v, \_) \neq 0 \in \mathrm{Hom}(V, \mathbb{R})$ for any non-zero $v \in V$. If $V$ is finite dimensional and $V$ admits a linear symplectic form, prove that $n = \dim(V)$ is necessarily even, say $n = 2m$.

(iii) Prove that $\omega \in \bigwedge^2 V^*$ is non-degenerate if and only if $\omega^m \neq 0$ in $\bigwedge^n V^*$.

*Solution.* That $A^k(V) \cong (V^{\wedge k})^*$ and $L^k(V) = (V^{\otimes k})^*$ is just a restatement of the universal properties of the exterior and tensor products. The universal property also imples that the diagram

$$\begin{array}{ccc} A^k(V) & \longrightarrow & (V^{\wedge k})^* \\ \downarrow & & \downarrow \\ L^k(V) & \longrightarrow & (V^{\otimes k})^* \end{array}$$

commutes. Let $\{v_1, \ldots, v_n\}$ be a basis for $V$ with dual basis $\{v_1^*, \ldots, v_n^*\}$ for $V^*$. Define a linear function $f\colon (V^*)^{\otimes k} \longrightarrow (V^{\otimes k})^*$ by

$$f_1 \otimes \cdots \otimes f_k \longmapsto (v_1 \otimes \cdots \otimes v_k \longmapsto f_1(v_1) \cdots f_k(v_k) \in \mathbb{R}).$$

We compute

$$f(v_{i_1}^* \otimes \cdots \otimes v_{i_k}^*)(v_{j_1} \otimes \cdots \otimes v_{j_k}) = v_{i_1}^*(v_{j_1}) \cdots v_{i_k}^*(v_{j_k}) = \delta_{(i_1 \ldots i_k),(j_1 \ldots j_k)}.$$

That is, $f$ maps a basis of $(V^*)^{\otimes k}$ to a basis of $(V^{\otimes k})^*$ and therefore is an isomorphism.

Similarly, define a linear function $g\colon (V^*)^{\wedge k} \longrightarrow (V^{\wedge k})^*$ by

$$f_1 \wedge \cdots \wedge f_k \longmapsto \Big(v_1 \wedge \cdots \wedge v_k \longmapsto \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) f_1(v_{\sigma(1)}) \cdots f_k(v_{\sigma(k)})\Big).$$

Observe that this indeed defines a linear map by the universal property of the exterior product. We again check on a basis that

$$g(v_{i_1}^* \wedge \cdots \wedge v_{i_k}^*)(v_{j_1} \wedge \cdots \wedge v_{j_k}) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) v_{i_1}^*(v_{\sigma(j_1)}) \cdots v_{i_k}^*(v_{\sigma(j_k)}) = \delta_{(i_1 \dots i_k),(j_1 \dots j_k)}$$

where $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. Again, we see that $g$ sends a basis of $(V^*)^{\wedge k}$ to a basis of $(V^{\wedge k})^*$. Furthermore, these calculations show that the diagram

$$\begin{array}{ccc} (V^{\wedge k})^* & \longleftarrow & (V^*)^{\wedge k} \\ \downarrow & & \uparrow \\ (V^{\otimes k})^* & \longleftarrow & (V^*)^{\otimes k} \end{array}$$

commutes.

Now, let $V$ be a vector space with a linear symplectic form $\omega$. We claim that there are $e, f \in V$ such that $\omega(e,f) = 1$ and a direct sum decomposition $V = \langle e, f \rangle \oplus W$ such that $\omega(v, w) = 0$ whenever $v \in \langle e, f \rangle$ and $w \in W$. Furthermore, $\omega$ restricts to a linear symplectic form on $W$.

To see this, pick $e \in V$ to be any nonzero vector. Since $\omega$ is nondegenerate, there is some $f \in V$ such that $\omega(e, f) \neq 0$ and we can renormalize $f$ and assume $\omega(e, f) = 1$. Define

$$W = \langle e, f \rangle^{\omega} = \{w \in V : \omega(e, w) = \omega(f, w) = 0\}.$$

Then $\langle e, f \rangle \cap W = 0$: for any $v = ae + bf \in \langle e, f \rangle \cap W$ we would have $0 = \omega(v, e) = -b$ and $0 = \omega(v, f) = a$ and therefore $v = 0$. Furthermore, $\langle e, f \rangle + W = V$: for any $v \in V$ we have

$$v = (-\omega(v, e)f + \omega(v, f)e) + (v + \omega(v, e)f - \omega(v, f)e)$$

where $-\omega(v, e)f + \omega(v, f)e \in \langle e, f \rangle$ and $v + \omega(v, e)f - \omega(v, f)e \in \langle e, f \rangle^{\omega}$.

Now, by induction $V$ admits a decomposition $V = W_1 \oplus \cdots \oplus W_m$ with $W_i = \langle e_i, f_i \rangle$ and $\omega(e_i, f_j) = \delta_{ij}$. In particular, it follows that $\dim(V) = 2m$ is even. Furthermore,

$$\omega^m(e_1, f_1, \dots, e_m, f_m) = 2^m \omega(e_1, f_1) \cdots \omega(e_m, f_m) = 2^m,$$

i. e. $\omega \neq 0$. Conversely, if $\omega \in (V^*)^{\wedge 2}$ is a 2−form such that $\omega^m \neq 0$, assume there were some $v \in V$ such that $\omega(v, \_) = 0$. Extend $v$ to a basis $\{v_1, v_2, \dots, v_{2m}\}$ of $V$. Then

$$\omega^m(v \wedge v_2 \wedge \cdots \wedge v_{2m}) = 0$$

even though $v \wedge v_2 \wedge \cdots \wedge v_{2m} \neq 0 \in V^{\wedge 2m}$. We conclude that $\omega(v, \_) \neq 0$ and therefore that $\omega$ is nondegenerate.

Exercise 5.8. Give a careful construction of the exterior differentiation operator $\mathrm{d} \colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ using local coordinates; show that this definition is independent of local coordinates and is well-defined.

*Solution.* Let $\alpha \in \Omega^k(M)$ and $p \in M$. Choose local coordinates $\{x^1, \dots, x^n\}$ in a neighborhood $U$ of $p$ and write

$$\alpha|_U = \sum_{|I|=k} f_I \,\mathrm{d}x^I$$

where the summation index denotes totally ordered subsets $I \subset \{1, \dots, n\}$ of cardinality $k$ and the $f_I$ are smooth functions. We define

$$\mathrm{d}\alpha|_U = \sum_{|I|=k} \sum_i \frac{\partial f_I}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^I.$$

Now suppose that $\{y^i\}$ are other local coordinates in a neighbourhood $V$ of $p$. To see that our definition is independent of coordinates we need to check that

$$\sum_{|I|=k}\sum_i \frac{\partial g_I}{\partial y^i}\,\mathrm{d}y^i\wedge \mathrm{d}y^I = \sum_{|I|=k}\sum_i \frac{\partial f_I}{\partial x^i}\,\mathrm{d}x^i\wedge \mathrm{d}x^I$$

as $(k+1)$–forms on $U\cap V$, where $\alpha|_V = \sum_{|I|=k} g_I\,\mathrm{d}y^I$. Let $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the transition map from $\{x^i\}$ to $\{y^i\}$, i. e. we have $y^i(p) = \phi^i(x^1(p),\dots,x^n(p))$ for $p\in U\cap V$. As usual, we then have

$$f_I = \sum_{|\mathcal{J}|=k} g_{\mathcal{J}}\frac{\partial \phi^I}{\partial x^{\mathcal{J}}}$$

and

$$\sum \frac{\partial f_I}{x^i}\,\mathrm{d}x^i\wedge \mathrm{d}x^I \overset{(*)}{=} \sum \frac{\partial g_{\mathcal{J}}}{\partial y^j}\frac{\partial y^j}{\partial x^i}\frac{\partial \phi^{\mathcal{J}}}{\partial x^I}\,\mathrm{d}x^i\wedge \mathrm{d}x^I = \sum \frac{\partial g_{\mathcal{J}}}{\partial y^j}\,\mathrm{d}y^j\wedge \mathrm{d}y^{\mathcal{J}}$$

as required. The equality $(*)$ follows from the commutativity of iterated partial derivatives with respect to coordinate bases. Having proved that our definition of the exterior derivative is independent of local coordinates, it follows as usual that the locally defined forms $\mathrm{d}\alpha|_U$ patch together into a global $(k+1)$–form $\mathrm{d}\alpha$.

Exercise 5.9. Let $M$ be a manifold. Prove that d satisfies the formula $\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^k\alpha\wedge \mathrm{d}\beta$ for $\alpha\in\Omega^k(M)$ and $\beta\in\Omega^\ell(M)$.
*Solution.* By the previous exercise we can compute in local coordinates $\{x^i\}$ and by linearity we can assume that $\alpha = f\,\mathrm{d}x^I$ and $\beta = g\,\mathrm{d}x^{\mathcal{J}}$ for some multi-indices $I$ and $\mathcal{J}$. Then we have

$$\begin{aligned}\mathrm{d}(\alpha\wedge\beta) &= \mathrm{d}(fg)\wedge \mathrm{d}x^I\wedge \mathrm{d}x^{\mathcal{J}} = g\,\mathrm{d}f\wedge \mathrm{d}x^I\wedge \mathrm{d}x^{\mathcal{J}} + f\,\mathrm{d}g\wedge \mathrm{d}x^I\wedge \mathrm{d}x^{\mathcal{J}} =\\ &= \sum_i g\frac{\partial f}{\partial x^i}\,\mathrm{d}x^i\wedge \mathrm{d}x^I\wedge \mathrm{d}x^{\mathcal{J}} + \sum_i f\frac{\partial g}{\partial x^i}\,\mathrm{d}x^i\wedge \mathrm{d}x^I\wedge \mathrm{d}x^{\mathcal{J}} =\\ &= \mathrm{d}\alpha\wedge\beta + (-1)^{|I|}\alpha\wedge \mathrm{d}\beta.\end{aligned}$$

Exercise 5.10. Prove that d commutes with pullback; that is $\mathrm{d}\circ f^* = f^*\circ \mathrm{d}$ for any smooth $f\colon M\longrightarrow N$.
*Solution.* Again, we can compute in local coordinates $\{x^i\}$ for $N$ and $\{y^i\}$ for $M$ and take some $k$–form $g\,\mathrm{d}x^I$ on $N$. Write $x^j\circ f = f^j$. Then

$$f^*(g\,\mathrm{d}x^I) = (g\circ f)\,\mathrm{d}f^I = \sum (g\circ f)\frac{\partial f^I}{\partial y^{\mathcal{J}}}\,\mathrm{d}y^{\mathcal{J}}.$$

Therefore

$$f^*(\mathrm{d}g\wedge \mathrm{d}x^I) = \sum\left(\frac{\partial g}{\partial x^i}\circ f\right)\mathrm{d}f^i\wedge \mathrm{d}f^I = \mathrm{d}(g\circ f)\wedge \mathrm{d}f^I = \mathrm{d}(f^*(g\,\mathrm{d}x^I)).$$