Math 535a Homework 6

Due Friday, March 31, 2017 by 5 pm

Please remember to write down your name on your assignment.

1. **The star operator and the cross product.** (double weight)
   (a) Let V be a vector space over $\mathbb{R}$ with an inner product $\langle -, - \rangle : V \times V \to \mathbb{R}$[1] Extend this inner prodct to the exterior algebra $\Lambda^\bullet V$ as follows: define

$$\langle v_1 \wedge \cdots \wedge v_s, w_1 \wedge \cdots \wedge w_t \rangle := \begin{cases} 0 & s \neq t \\ \det(\langle v_i, w_j \rangle) & s = t \end{cases}$$

The second expression means: take the determinant of the linear transformation associated to the $s \times s$ matrix with entries $\langle v_i, w_j \rangle$. Check that this gives a well-defined symmetric bilinear map. Moreover, show that if $e_1, \ldots, e_n$ is an any orthonormal basis for V, then $\{e_{i_1} \wedge \cdots \wedge e_{i_k} | k \leq n, i_1 < \cdots < i_k\}$ is an orthonormal basis for $\Lambda^\bullet V$. (Recall that an orthonormal basis is a basis satisfying $\langle e_i, e_j \rangle = 1$ if $i = j$ and 0 if $i \neq j$).

Remark: In this class, we defined the determinant of a linear map $T : V \to V$ to be the scalar d such that $T^{\wedge \dim V} : \wedge^{\dim V} V \to \wedge^{\dim V} V$ is multiplication by d. You may use without justification the following standard formula for the determinant: if T has matrix A with respect to a given basis, then

$$\det A = \sum_\pi (\text{sgn } \pi) a_{1\pi(1)} \cdots a_{n\pi(n)}$$

where π runs over all permutations of $\{1, \ldots, n\}$ and sgn π denotes the sign of the permutation π. It is not difficult to prove, by induction on n or directly, that this definition coincides with the definition we gave.

   (b) Let V be a vector space of dimension n. Recall that in class we defined an *orientation of* V to be a choice of connected component of the topological space $\Lambda^n V \backslash \{0\}$ (or equivalently an element of the set $or(V) := \Lambda^n V \backslash \{0\} / \mathbb{R}_+$ where $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$ acts by scaling).[2]

If V is an oriented vector space (meaning a vector space equipped with an orientation $\sigma \in or(V)$) which has an inner product $\langle -, - \rangle$, then there is a linear transformation, called the *Hodge star operator*,

$$\star : \Lambda^\bullet V \to \Lambda^\bullet V.$$

[1]Recall that an inner product is a map $V \times V \to V$ which is symmetric, bilinear, and non-degenerate. Moreover, such inner products induce norms $||v|| := \langle v, v \rangle$ and metrics $d(v, w) = ||v - w||$.

[2]Equivalently, one defines $or(V)$ to be the set of connected components of $\Lambda^n V \backslash 0$.

To define it, note first that there is a unique non-zero vector $\omega \in \Lambda^{\dim V} V$ with $||\omega|| = \sqrt{\langle \omega, \omega \rangle} = 1$ lying in the component determined by the orientation σ (meaning that $[\omega] = \sigma$) [3]. Call this element ω the *volume form* V induced by the orientation and inner product.

Now, the element ω induces a linear maps $vol_\omega : \Lambda^n V \to \mathbb{R}$ and $vol_\omega : \Lambda^\bullet V \to \mathbb{R}$ sending ω to 1 and all other degree k wedges ($k < n$ to zero) (when restricted to $\Lambda^n V$, vol_ω is an isomorphism). Hence, we get a map

$$\Lambda^\bullet V \to (\Lambda^\bullet V)^*$$

sending α to the functional $vol_\omega(\alpha \wedge -)$. Finally, using $\langle -, - \rangle$, one identifies $(\Lambda^\bullet V)^* \cong \Lambda^\bullet V$. Define $\star$ to be the isomorphism induced by the composition

$$\star : \Lambda^\bullet V \to (\Lambda^\bullet V)^* \to \Lambda^\bullet V.$$

Observe that $\star$ restricts to a maps $\Lambda^k V \to \Lambda^{n-k} V$ for each k, where $n = \dim(V)$.

Prove that on $\Lambda^p V$,

$$\star\star = (-1)^{p(n-p)}.$$

Hint: It suffices to check this on any orthonormal basis of $\Lambda^k V$, for instance one induced by an orthonormal basis of V.

(c) Prove that for arbitrary $v, w \in \Lambda^p V$, their inner product is given by

$$\langle v, w \rangle = \star(w \wedge \star v) = \star(v \wedge \star w) = \langle \star v, \star w \rangle.$$

(in particular, $\star : \Lambda^k V \to \Lambda^{n-k} V$ is an isometry).

(d) Let $V = \mathbb{R}^3$, equipped with its standard Euclidean inner product; let e_1, e_2, e_3 denote the standard basis. Pick orientation on V determined by the volume element $e_1 \wedge e_2 \wedge e_3$; this determines a Hodge star map $\star$ as above. Compare the map

$$\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\wedge} \Lambda^2 \mathbb{R}^3 \xrightarrow{\star} \mathbb{R}^3$$

to the cross product of vectors in the usual sense.

2. Let V be a finite-dimensional vector space, and let $\xi \in V$. Prove that the composition

$$\Lambda^p V \xrightarrow{\xi \wedge -} \Lambda^{p+1} V \xrightarrow{\xi \wedge -} \Lambda^{p+2} V$$

is an exact sequence. That is, the image of the first map is the kernel of the second map.

3. Use the Mayer-Vietoris sequence to prove that

$$H^k_{dR}(S^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}.$$

[3] Note that for $n = \dim(V)$ since the vector space $\Lambda^n V$ is one-dimensional, it possesses only n-wedges α with $\langle \alpha, \alpha \rangle = 1$

You may assume, as input, the computation of the de Rham cohomology of $\mathbb{R}^n$ and S^1. Inductively prove from there that

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

4. Suppose that $M = M_1 \coprod M_2$. Then prove that
$$H^k_{dR}(M) = H^k_{dR}(M_1) \oplus H^k_{dR}(M_2).$$

5. (double weight) Complete the proof that a short exact sequence of co-chain maps
$$0 \to B^\bullet \to C^\bullet \to D^\bullet \to 0$$
gives rise to a long exact sequence on cohomology. (Your solution should be written carefully and completely).

6. (a) Let V be a finite-dimensional vector space admitting a direct sum decomposition $V \cong U \oplus W$. Prove that there is a canonical map $or(V) \times or(W) \to or(U)$. In other words, an orientation of V along with an orientation of W determines an orientation of the complementary subspace U.

   (b) Now let $X^d \subset \mathbb{R}^{d+1}$ be a d-dimensional submanifold of Euclidean space. Define the *normal bundle* of X to be the line bundle whose fiber at $p \in X$ is the orthogonal complement of T_pX in $T_p\mathbb{R}^{d+1} \cong \mathbb{R}^{d+1}$; that is, $NX = \{(p, v) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} | p \in X, v \in (T_pX)^\perp\}$ where we are using the standard Euclidean inner product on $\mathbb{R}^{d+1}$ to take orthogonal complement.

   Show (without detailed proof, but constructing π and the local trivializations) that NX is in fact a line bundle over X. Then show that X is orientable if and only if NX has a nowhere vanishing section, also called a nowhere vanishing *normal field.*

   (**Remark**: More generally, recall that for $X \hookrightarrow Y$ an embedding we have a vector bundle over X $TY|_X$ and a sub-bundle $TX \subset TY|_X$. The *normal bundle* for $X \subset Y$ is by definition the *quotient vector bundle* $NX := TY|_X/TX$; that is, a vector bundle, constructed in much the same way as last week's homework, whose fiber at each point $p \in X$ is T_pY/T_pX. This definition is related to the one above in the same way that, in the presence of an inner product on V, if $U \subset V$, then $U^\perp \cong V/U$.

7. Prove that real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd. (**Hint**: Observe that the antipodal map on the n-sphere S^n is orientation preserving if and only if n is odd. It may or may not be helpful to use the characterization that a connected manifold M is orientable iff it admits a nonvanishing top form iff the vector bundle $\Lambda^{\dim M} TM$ is trivial. The previous problem may also help construct and analyze orientations on S^n.)