Homework 6

Exercise 6.1.

(i) Let V be a vector space over $\mathbb{R}$ with an inner product $\langle \_,\_\rangle\colon V\times V\longrightarrow \mathbb{R}$. Extend this inner product to the exterior algebra $\bigwedge^\bullet V$ by setting

$$\langle v_1\wedge\cdots\wedge v_s, w_1\wedge\cdots\wedge w_t\rangle := \begin{cases} 0 & s\neq t \\ \det(\langle v_i, w_j\rangle) & s=t. \end{cases}$$

Check that this gives a well-defined symmetric bilinear map. Moreover show that if $e_1,\dots,e_n$ is an orthonormal basis for V, then $\{e_{i_1}\wedge\cdots\wedge e_{i_k} : k\leq n, i_1<\cdots<i_k\}$ is an orthonormal basis for $\bigwedge^\bullet V$.

(ii) Let V be a vector space of dimension n. Recall that in class we defined an *orientation of* V to be a choice of connected component of the topological space $\bigwedge^n V\smallsetminus\{0\}$. If V is an oriented vector space which has an inner product $\langle \_,\_\rangle$, then there is a linear transformation, called the *Hodge star operator*,

$$\star\colon \textstyle\bigwedge^\bullet V\longrightarrow \bigwedge^\bullet V.$$

To define it, note first that there is a unique non-zero vector $\omega\in\bigwedge^{\mathrm{top}}V$ with $\|\omega\| = \sqrt{\langle\omega,\omega\rangle}=1$ lying in the component determined by the orientation σ. Call this element ω the *volume form of* V induced by the orientation and inner product.

Now, the element ω induces linear maps $\mathrm{vol}_\omega\colon \bigwedge^n V\longrightarrow\mathbb{R}$ and $\mathrm{vol}_\omega\colon\bigwedge^\bullet V\longrightarrow\mathbb{R}$ sending ω to 1 and all other degree k wedges to zero. Hence, we get a map

$$\textstyle\bigwedge^\bullet V\longrightarrow(\bigwedge^\bullet V)^*$$

sending α to the functional $\mathrm{vol}_\omega(\alpha\wedge\_)$. Finally, using $\langle\_,\_\rangle$ one identifies $(\bigwedge^\bullet V)^*\cong\bigwedge^\bullet V$. Define $\star$ to be the isomorphism induced by the composition

$$\star\colon\textstyle\bigwedge^\bullet V\longrightarrow(\bigwedge^\bullet V)^*\longrightarrow\bigwedge^\bullet V.$$

Observe that $\star$ restricts to maps $\bigwedge^k V\longrightarrow\bigwedge^{n-k}V$ for each k, where $n=\dim(V)$. Prove that, on $\bigwedge^p V$,

$$\star\star = (-1)^{p(n-p)}.$$

(iii) Prove that for arbitrary $v,w\in\bigwedge^p V$, their inner product is given by

$$\langle v,w\rangle = \star(w\wedge\star v) = \star(v\wedge\star w) = \langle\star v,\star w\rangle.$$

(iv) Let $V=\mathbb{R}^3$ equipped with its standard Euclidean inner product. Let $\{e_1,e_2,e_3\}$ denote the standard basis. Pick an the orientation on V determined by the volume element $e_1\wedge e_2\wedge e_3$. This determines a Hodge star map $\star$ as above. Compare the map

$$\mathbb{R}^3\times\mathbb{R}^3\xrightarrow{\ \wedge\ }\textstyle\bigwedge^2\mathbb{R}\xrightarrow{\ \star\ }\mathbb{R}^3$$

to the cross product of vectors in the usual sense.

Solution.

(i) The map $\varphi\colon V^{\times k}\times V^{\times k}\longrightarrow\mathbb{R}^{k\times k}$ with $\varphi((v_i),(w_j)) = (\langle v_i,w_j\rangle)_{ij}$ is multilinear and evidently satisfies $\varphi((w_j),(v_i)) = \varphi((v_i),(w_j))^T$. The composition $\psi=\det\circ\varphi$ therefore is symmetric under interchanging (v_i) and (w_j) and for fixed (w_j) or fixed (v_i) respectively it is an alternating map because det is. It follows that φ descends to the symmetric bilinear form $\bigwedge^\bullet V\times\bigwedge^\bullet V\longrightarrow\mathbb{R}$ described in the question. Consider the scalar product

$$\langle e_{i_1}\wedge\cdots\wedge e_{i_k}, e_{j_1}\wedge\cdots\wedge e_{j_k}\rangle = \det(\langle e_{i_s},e_{j_t}\rangle)_{st} = \det(\delta_{i_s,j_t})_{st}.$$

The Leibniz formula for the determinant implies that this is equal to

$$\sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\delta_{i_1,j_{\sigma(1)}} \cdots \delta_{i_k,j_{\sigma(k)}} = \delta_{i_1,j_1} \cdots \delta_{i_k,j_k}$$

because $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. That is, $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : k \leq n, i_1 < \cdots < i_k\}$ is an orthonormal basis for $\bigwedge^\bullet V$. The existence of this orthonormal basis immediately implies that $\langle \_, \_ \rangle$ is non-degenerate.

(ii) The set $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : i_1 < \cdots < i_p\}$ is a basis of $\bigwedge^p V$. Let $i_{p+1} < \cdots < i_n$ be such that $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$. Then $\operatorname{vol}_\omega(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{i_{p+1}} \wedge \ldots e_{i_n}) = \pm 1$. We conclude that

$$\star(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \pm e_{i_{p+1}} \wedge \cdots \wedge e_{i_n}.$$

That is, up to a sign, $\star$ sends an orthonormal basis to an orthonormal basis. Therefore $\star$ is an isometry. Let $\alpha, \beta \in \bigwedge^p V$. Then

$$\langle \alpha, \beta \rangle = \langle \star\beta, \star\alpha \rangle = \operatorname{vol}_\omega(\beta \wedge \star\alpha) = (-1)^{p(n-p)} \operatorname{vol}_\omega(\star\alpha \wedge \beta) = (-1)^{p(n-p)} \langle \star\star\alpha, \beta \rangle.$$

Since $\langle \_, \_ \rangle$ is non-degenerate, it follows that $\star\star = (-1)^{p(n-p)}$ on $\bigwedge^p V$.

(iii) Using $\star\omega = 1$ and $\star 1 = \omega$ we compute

$$\begin{aligned} \langle v, w \rangle \overset{\text{(ii)}}{=} \langle \star v, \star w \rangle &= \star(\langle \star v, \star w \rangle \omega) = \star(v \wedge \star w) \\ &= \star(\langle \star w, \star v \rangle \omega) = \star(w \wedge \star v). \end{aligned}$$

(iv) Using our arguments from (ii) we have

$$\begin{aligned} \star(e_1 \wedge e_2) &= e_3 = e_1 \times e_2 \\ \star(e_2 \wedge e_3) &= e_1 = e_2 \times e_3 \\ \star(e_1 \wedge e_3) &= -e_2 = e_1 \times e_3. \end{aligned}$$

Exercise 6.2. Let V be a finite-dimensional vector space, and let $\xi \in V$. Prove that

$$\textstyle\bigwedge^p V \xrightarrow{\xi \wedge \_} \bigwedge^{p+1} V \xrightarrow{\xi \wedge \_} \bigwedge^{p+2} V$$

is an exact sequence.

Solution. First, for any $\alpha \in \bigwedge^p V$ we have $\xi \wedge \xi \wedge \alpha = 0$ because $\xi \wedge \xi = -\xi \wedge \xi$. It follows that the given sequence is at least a complex. Now let $\beta \in \bigwedge^{p+1} V$ be such that $\xi \wedge \beta = 0 \in \bigwedge^{p+2} V$. Let $\xi_1 = \xi$ and extend ξ_1 to a basis $\{\xi_1, \ldots, \xi_n\}$ of V. Write

$$\omega = \sum_{1 \leq i_1 < \cdots < i_p \leq n} c_I \cdot \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}$$

where I denotes the multiindex $(i_1, \ldots, i_p)$ as usual. Then

$$0 = \xi \wedge \omega = \sum_{2 \leq i_1 < \cdots < i_p \leq n} c_I \cdot \xi \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}$$

because $\xi = \xi_1$ and therefore $\xi \wedge \xi_1 = 0$. Since the $\xi \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}$ appearing on the right hand side are linearly independent we must have $c_I = 0$ whenever $i_1 \geq 2$. Consequently,

$$\omega = \sum_{\substack{1 \leq i_1 < \cdots < i_p \leq n \\ i_1 = 1}} c_I \xi \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_p} = \xi \wedge \sum_{\substack{1 \leq i_1 < \cdots < i_p \leq n \\ i_1 = 1}} c_i \xi_{i_2} \wedge \cdots \wedge \xi_{i_p} \in \operatorname{im}(\xi \wedge \_).$$

Exercise 6.3. Use the Mayer–Vietoris sequence to prove that

$$H^k_{\mathrm{dR}}(S^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Inductively, prove from there that

$$H^k_{\mathrm{dR}}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Let $N \in S^n$ and $S \in S^n$ be the north and south pole respectively. Set $U_1 = S^n \setminus \{N\}$ and $U_2 = S^n \setminus \{S\}$. Then $U_1 \cup U_2 = S^n$ and $U_1 \cap U_2 \simeq S^{n-1}$. The Mayer–Vietoris sequence starts

$$0 \longrightarrow H^0_{\mathrm{dR}}(S^n) \longrightarrow H^0_{\mathrm{dR}}(U_1) \oplus H^0_{\mathrm{dR}}(U_2) \longrightarrow H^0_{\mathrm{dR}}(S^{n-1}) \longrightarrow H^1_{\mathrm{dR}}(S^n)$$

and a general term looks like

$$\ldots \longrightarrow H^{k-1}_{\mathrm{dR}}(S^{n-1}) \longrightarrow H^k_{\mathrm{dR}}(S^n) \longrightarrow H^k_{\mathrm{dR}}(U_1) \oplus H^k_{\mathrm{dR}}(U_2) \longrightarrow H^k_{\mathrm{dR}}(S^{n-1}) \longrightarrow H^{k+1}_{\mathrm{dR}}(S^n) \longrightarrow \ldots$$

Now, $U_i \simeq *$ and therefore $H^0_{\mathrm{dR}}(U_i) = \mathbb{R}$ and $H^k_{\mathrm{dR}}(U_i) = 0$ for $k \geq 1$. It follows that $H^k_{\mathrm{dR}}(S^n) \cong H^{k-1}_{\mathrm{dR}}(S^{n-1})$ for $k \geq 2$. Furthermore, S^n is connected, so $H^0_{\mathrm{dR}}(S^n) = \mathbb{R}$. Looking at the start of the Mayer–Vietoris sequence, we see that the kernel of $H^0_{\mathrm{dR}}(U_1) \oplus H^0_{\mathrm{dR}}(U_2) \longrightarrow H^0_{\mathrm{dR}}(S^{n-1})$ has dimension 1. Therefore, its image must have dimension 1 as well which implies that the boundary map $H^0_{\mathrm{dR}}(S^{n-1}) \longrightarrow H^1_{\mathrm{dR}}(S^n)$ is the zero map. Since it is also surjective because $H^1_{\mathrm{dR}}(U_i) = 0$ we conclude that $H^1_{\mathrm{dR}}(S^n) = 0$.

Inductively combining this with our knowledge of $H^\bullet_{\mathrm{dR}}(S^1)$ we get precisely

$$H^k_{\mathrm{dR}}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 6.4. Suppose that $M = M_1 \sqcup M_2$. Prove that then $H^k_{\mathrm{dR}}(M) = H^k_{\mathrm{dR}}(M_1) \oplus H^k_{\mathrm{dR}}(M_2)$.
Solution. Using $H^k_{\mathrm{dR}}(M_1 \cap M_2) = 0$ for all k, the Mayer–Vietoris sequence for $M = M_1 \cup M_2$ starts with

$$0 \longrightarrow H^0_{\mathrm{dR}}(M) \longrightarrow H^0_{\mathrm{dR}}(M_1) \oplus H^0_{\mathrm{dR}}(M_2) \longrightarrow H^0_{\mathrm{dR}}(M_1 \cap M_2) = 0$$

and a general term looks like

$$0 = H^{k-1}_{\mathrm{dR}}(M_1 \cap M_2) \longrightarrow H^k_{\mathrm{dR}}(M) \longrightarrow H^k_{\mathrm{dR}}(M_1) \oplus H^k_{\mathrm{dR}}(M_2) \longrightarrow H^k_{\mathrm{dR}}(M_1 \cap M_2) = 0.$$

The claim follows.

Exercise 6.5. Complete the proof that a short exact sequence of cochain maps

$$0 \longrightarrow B^\bullet \xrightarrow{f^\bullet} C^\bullet \xrightarrow{g^\bullet} D^\bullet \longrightarrow 0$$

gives rise to a long exact sequence on cohomology.
Solution. The exact sequence gives rise to a commutative diagram

$$\begin{array}{ccccccc}
 & B^n/\mathrm{im}(d_B^{n-1}) & \xrightarrow{f^n} & C^n/\mathrm{im}(d_C^{n+1}) & \xrightarrow{g^n} & D^n/\mathrm{im}(d_D^{n+1}) & \longrightarrow 0 \\
 & \downarrow d_B^n & & \downarrow d_C^n & & \downarrow d_D^n & \\
0 \longrightarrow & \ker(d_B^{n+1}) & \xrightarrow[f^{n+1}]{} & \ker(d_C^{n+1}) & \xrightarrow[g^{n+1}]{} & \ker(d_D^{n+1}) &
\end{array}$$

with exact rows. Note that the kernels of the vertical maps are the degree n cohomologies of $B^\bullet$, $C^\bullet$ and $D^\bullet$ and the cokernels are the degree $n+1$ cohomologies respectively. Any diagram of this shape immediately induces exact sequences

$$H^n(B) \longrightarrow H^n(C) \longrightarrow H^n(D)$$

and

$$H^{n+1}(B) \longrightarrow H^{n+1}(C) \longrightarrow H^{n+1}(D).$$

What is left to check is that there is a connecting homomorphism $\delta\colon H^n(D) \longrightarrow H^{n+1}(B)$ with corresponding exactness at $H^n(D)$ and $H^{n+1}(B)$.

To construct δ, let $[\alpha] \in H^n(D)$. Since $C^n/\mathrm{im}(d_C^{n-1}) \longrightarrow D^n/\mathrm{im}(d_D^{n-1})$ is surjective, there is some $\beta \in C^n$ such that $[\beta] \longmapsto [\alpha]$ along g. Then, $g^{n+1}(d\beta) = dg^n(\beta) = d\alpha = 0$ and so there is some $\gamma \in \ker(d_B^{n+1})$ such that $\gamma \longmapsto d\beta$ along f. We want to define $\delta([\alpha]) = [\gamma]$. But for this to make sense, we should first check that $[\gamma]$ does not depend on all the choices made so far. So, let $\alpha' = \alpha + d\xi$ be such that $[\alpha'] = [\alpha]$. Choose some $\beta' \in C^n$ such that $[\beta'] \longmapsto [\alpha']$ and $\gamma' \in \ker(d_B^{n+1})$ such that $\gamma' \longmapsto d\beta'$. Then $g([\beta'] - [\beta]) = [\alpha' - \alpha] = [d\xi] = 0$ and therefore there is some $\zeta \in B^n$ such that $f([\zeta]) = [\beta' - \beta]$. We find that $f(\gamma' - \gamma) = d\beta' - d\beta = df([\zeta]) = f(d\zeta)$. But f is injective, so $\gamma' - \gamma = d\zeta$ and therefore $[\gamma'] = [\gamma] \in H^{n+1}(B)$. We conclude that the definition $\delta([\alpha]) = [\gamma]$ makes sense and gives a connecting homomorphism $H^n(D) \longrightarrow H^{n+1}(B)$.

To check exactness at $H^n(D)$ first take some $[\beta] \in H^n(C)$. Then to calculate $\delta(g([\beta]))$ we just need to observe that $0 \longmapsto 0 = d\beta$. So $\delta(g([\beta])) = 0$. Conversely, let $[\alpha] \in H^n(D)$ with $\delta([\alpha]) = 0$. Choose some $\beta \in C^n$ such that $g([\beta]) = [\alpha]$. Because $[\alpha] \in \ker\delta$ we must have $d\beta = f(d\zeta)$ for some $\zeta \in B^n$. Observe that this means $d(\beta - f(\zeta)) = 0$ and $g([\beta - f(\zeta)]) = g([\beta]) = [\alpha]$. Hence, $[\beta - f(\zeta)]$ is a preimage of $[\alpha]$ in $H^n(C)$ and we conclude the exactness at $H^n(D)$.

Now, to check exactness at $H^{n+1}(C)$ first take some $[\alpha] \in H^n(D)$. Pick $\beta \in C^n$ with $g([\beta]) = [\alpha]$ and $\gamma \in B^{n+1}$ with $f(\gamma) = d\beta$. Then $f(\delta([\alpha])) = f([\gamma]) = [d\beta] = 0$. Conversely, let $[\gamma] \in H^{n+1}(B)$ with $f([\gamma]) = 0 \in H^{n+1}(C)$, say $f(\gamma) = d\beta$ for some $\beta \in C^n$. But then we have $\delta(g([\beta])) = [\gamma]$ by the definition of δ. So we also have exactness at $H^{n+1}(C)$.

Exercise 6.6.

(i) Let V be a finite-dimensional vector space admitting a direct sum decomposition $V \cong U \oplus W$. Prove that there is a canonical map $\mathrm{or}(V) \times \mathrm{or}(W) \longrightarrow \mathrm{or}(U)$. In other words, an orientation of V along with an orientation on W determines an orientation of the complementary subspace U.

(ii) Now let $X^d \subset \mathbb{R}^{d+1}$ be a d–dimensional submanifold of Euclidean space. Define the *normal bundle* of X to be the line bundle whose fiber at $p \in X$ is the orthogonal complement of T_pX in $T_p\mathbb{R}^{d+1} \cong \mathbb{R}^{d+1}$. That is, $NX = \{(p, v) : p \in X, v \in (T_pX)^\perp\}$ where we are using the standard Euclidean inner product on $\mathbb{R}^{d+1}$ to take the orthogonal complement.
Show that NX is in fact a line bundle over X. Then show that X is orientable if and only if NX has a nowhere vanishing section, also called a nowhere vanishing *normal field.*

Solution.

(i) Assume given an orientation of V and an orientation of W. Let $\{w_1, \dots, w_k\}$ be an oriented basis of W. Given a basis $\{u_1, \dots, u_\ell\}$ of U, say that it is positively oriented if and only if the basis $\{w_1, \dots, w_k, u_1, \dots, u_\ell\}$ of V is positively oriented. This gives a map $\mathrm{or}(V) \times \mathrm{or}(W) \longrightarrow \mathrm{or}(U)$.

(ii) To see that NX is a line bundle over X let $p \in X$ and choose a submanifold chart for X around p. That is, let $U \subset \mathbb{R}^{d+1}$ be open, $p \in U$ and assume there is a diffeomorphism $\varphi\colon U \longrightarrow \mathbb{R}^{d+1}$ such that $\varphi(p) = 0$ and $\varphi(X \cap U) \subset \mathbb{R}^d \times \{0\}$. The map $\psi\colon NX|_{U\cap X} \longrightarrow \varphi(X \cap U) \times (\{0\} \times \mathbb{R})$ with $\psi(x, v) = (\varphi(x), \pi_n(\mathrm{d}\varphi_x(v)))$ will be a vector bundle chart for NX over U where $\pi_n(x_1, \dots, x_n) = x_n$ is the projection.
Suppose NX has a nowhere vanishing section ν. Let $\Omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{d+1}$ be the standard volume of $\mathbb{R}^{d+1}$ and define $\omega = \nu \lrcorner \Omega|_X$. Then ω vanishes on vectors in NX and therefore defines a d–form on X.

Because ν was nowhere vanishing and Ω is a volume form on $\mathbb{R}^{d+1}$ the contraction $\nu \lrcorner \Omega|_X$ is nowhere vanishing on TX and therefore a volume form on X. In particular, it defines an orientation of X.

Conversely, suppose X admits a volume form $\omega \in \Omega^d(X)$. Define a 1–form $\psi\colon TY|_X \longrightarrow X \times \mathbb{R}$ by $\psi(v)\omega = v \lrcorner \Omega$ on sections. Because Ω and ω are nowhere vanishing this is a well defined map and descends to a bundle isomorphism

$$NX \cong TY|_X/TX \xrightarrow{\sim} X \times \mathbb{R}.$$

The preimage of the constant section 1 will be a nowhere vanishing normal field on X.

Exercise 6.7. Prove that real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd.

Solution. Consider the radial vector field $\nu = r\,\partial_r = \frac{1}{2}\nabla r^2$ where $r^2 = x_1^2 + \cdots + x_{n+1}^2$ on $\mathbb{R}^{n+1}$. The restriction of ν to $S^n \subset \mathbb{R}^{n+1}$ is just the outward pointing unit normal field on S^n. Consequently, the standard volume form on S^n is given by $\mathrm{dvol}_{S^n} = (\nu \lrcorner \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1})|_{S^n}$.

Let $\rho\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ be the inversion map $x \longmapsto -x$. Then, pulling back along ρ, we find $\rho^*(\nu) = \nu$ and $\rho^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1}) = (-1)^{n+1}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1}$. Therefore, ρ^* preserves dvol_{S^n} if and only if n is odd. Consequently, if n is odd, the volume form dvol_{S^n} descends to a volume form on $\mathbb{RP}^n = S^n/\{\pm 1\}$. That is, for odd n we have found an orientation on $\mathbb{RP}^n$.

Now, for even n, assume there were a volume form on $\mathbb{RP}^n$. Pulling back along the covering $S^n \longrightarrow \mathbb{RP}^n$, we can equivalently consider it to be a ρ^*–invariant volume form ω on S^n. But then there is a strictly positive function $f\colon S^n \longrightarrow \mathbb{R}$ such that $\omega = f\,\mathrm{dvol}_{S^n}$. We compute $\omega = \rho^*\omega = (f \circ \rho) \cdot \rho^*\mathrm{dvol}_{S^n} = -(f \circ \rho)\,\mathrm{dvol}_{S^n}$ since n is even. But then $f = -(f \circ \rho)$ which is impossible for $f > 0$. We conclude that $\mathbb{RP}^n$ cannot be orientable for n even.