Math 535a Homework 7

Due Wednesday, April 19, 2017 by 5 pm

Please remember to write down your name on your assignment.

1. Give a careful computation of the de Rham cohomology (and hence, the Euler characteristic) of a genus g surface Σ_g, using the sketch given in class or another method of your choice.

2. For the next problem, recall some basic definitions sketched in class: If M is a manifold of dimension n, an *oriented atlas* for M is a collection $\{(U_\alpha, \phi_\alpha, \sigma_\alpha)\}_{\alpha \in I}$ where (U_α, ϕ_α) is a chart, and $\sigma_\alpha \in or(\phi_\alpha(U_\alpha))$ is a choice of orientation of the orientable manifold $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$, such that each transition function $\phi_\alpha \circ \phi_\beta^{-1}$ is *orientation preserving*, meaning that it sends the choice of orientation σ_β to the choice of orientation σ_α.
   (a) Show that by modifying ϕ_α if needed, without loss of generality, an oriented atlas is equivalent (as atlases) to an oriented atlas such that each σ_α is the restriction of the standard orientation $[e_1 \wedge \cdots \wedge e_n]$ on $\mathbb{R}^n$; in that case, orientation preserving is equivalent to requiring that the derivative of each transition function be positive $d(\phi_\alpha \circ \phi_\beta^{-1}) \in GL^+(\mathbb{R}^n)$. (we will call such an atlas a *Euclidean oriented atlas.*)

   (b) We have seen that M is orientable if and only if it admits a Euclidean oriented atlas, in the sense above. Show that an orientation on M induces a unique maximal oriented atlas and vice versa. [1]

   (c) (*double weight compared to other sub-problems.*) Let M be an oriented manifold of dimension n and $\omega \in \Omega^n_c(M)$. If $\{(U_\alpha, \phi_\alpha, \sigma_\alpha)\}$ is an *oriented* atlas (which is locally finite), and $\{f_\alpha : M \to \mathbb{R}\}$ a partition of unity adapted to the cover $\{U_\alpha\}$, we defined
   $$\int_M \omega := \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(f_\alpha \omega).$$
   Show that this definition is well-defined and does not depend on the choice of oriented atlas induced by the orientation of M, or the choice of partition of unity.

3. (*The Poincaré Lemma for compactly supported de Rham cohomology – double weight problem*). The goal of this problem is to, in steps, compute that
   $$H^n_c(\mathbb{R}^m) = \begin{cases} 0 & m \neq n \\ \mathbb{R} & m = n. \end{cases}$$
   We will proceed in steps:
   (a) Argue why the result is true for $\mathbb{R}^0$ (this should be very short.)

[1]Two oriented atlases $\mathcal{A}_1 = \{(U_\alpha, \phi_\alpha, \sigma_\alpha\}_{\alpha \in I}$ and $\mathcal{A}_2 = \{(U_\beta, \phi_\beta, \sigma_\beta\}_{\beta \in J}$ are compatible if the transition functions between them $\phi_\alpha \circ \phi_\beta^{-1}$ preserve orientations. Using this notion of compatible, one defines the notion of a maximal oriented atlas associated to an oriented atlas $\mathcal{A}$ as the unique oriented atlas containing $\mathcal{A}$ and containing every oriented atlas compatible with $\mathcal{A}$.

(b) Show that any form $\omega \in \Omega^n_c(\mathbb{R}^m)$ can be written in a unique way as

$$\omega = \sum_{i_1<\cdots<i_{n-1}<m} f_{i_1,\ldots,i_{n-1}} dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}} \wedge dx_m + \sum_{j_1<\cdots<j_n<m} g_{j_1,\ldots,j_n} dx_{j_1} \wedge \cdots \wedge dx_{j_n}.$$

In terms of this decomposition, set

$$P(\omega) = \sum_{i_1<\cdots<i_{n-1}<m} \left(\int_{-\infty}^{+\infty} f_{i_1,\ldots,i_{n-1}} dx_m \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}}$$

Show that the linear map $P : \Omega^n_c(\mathbb{R}^m) \to \Omega^{n-1}_c(\mathbb{R}^{m-1})$ so defined induces a linear map $P_* : H^n(\mathbb{R}^m) \to H^{n-1}(\mathbb{R}^{m-1})$ defined by the property that $P_*([\omega]) = [P(\omega)]$.

(c) Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-1}$ be the projection to the first $m-1$ coordinates, defined by $\pi(x_1, \ldots, x_m) = (x_1, \ldots, x_{m-1})$. Pick your favorite compactly supported function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying $\int_{-\infty}^{\infty} \chi(t)dt = 1$. Show that the linear map $Q : \Omega^{n-1}_c(\mathbb{R}^{m-1}) \to \Omega^n_c(\mathbb{R}^m)$ defined by the property that

$$Q(\alpha) = \chi(x_m)\pi^*(\alpha) \wedge dx_m$$

induces a linear map $Q_* : H^{n-1}_c(\mathbb{R}^{m-1}) \to H^n_c(\mathbb{R}^m)$.

(d) Show that the composition $P \circ Q$ is the identity, so that $P_* \circ Q_*$ is also the identity.

(e) For ω decomposed as in part (a), set

$$\begin{aligned} K_n(\omega)(x_1, \ldots, x_m) &= \sum_{i_1<\cdots<i_{n-1}<m} \left(\int_{-\infty}^{x_m} \chi(t)dt \right) \left(\int_{-\infty}^{\infty} f_{i_1,\ldots,i_{n-1}}(x_1, \ldots, x_{m-1}, t)dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}} \\ &- \sum_{i_1<\cdots<i_{n-1}<m} \left(\int_{-\infty}^{x_m} f_{i_1,\ldots,i_{n-1}}(x_1, \ldots, x_{m-1}, t)dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}}; \end{aligned}$$

this defines a form $K_n(\omega) \in \Omega^{n-1}_c(\mathbb{R}^m)$. Show that

$$Q \circ P(\omega) - \omega = \pm(dK_n(\omega) - K_{n+1}(d\omega))$$

where $\pm$ depends on n.

(f) Conclude that $P_* : H^n(\mathbb{R}^m) \to H^{n-1}(\mathbb{R}^{m-1})$ is an isomorphism for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$.

(g) Conclude, using the previous part and part (a), that as desired,

$$H^n_c(\mathbb{R}^m) = \begin{cases} 0 & m \neq n \\ \mathbb{R} & m = n. \end{cases}$$

Conclude also that the map

$$\int_{\mathbb{R}^m} (-) : H_c^m(\mathbb{R}^m) \to \mathbb{R}$$

(using the standard orientation on $\mathbb{R}^m$) is an isomorphism. (*Hint*: compare this map to the composition of P_* isomorphisms $H_c^m(\mathbb{R}^m) \to H_c^{m-1}(\mathbb{R}^{m-1}) \to H_c^{m-2}(\mathbb{R}^{m-2}) \to \cdots \to H_c^0(\mathbb{R}^0) \cong \mathbb{R}$.)

Remark: An essentially identical argument (just done in local coordinates on M) to parts (b) - (f) also shows that $H_c^n(M \times \mathbb{R}) \cong H_c^{n-1}(M)$ for all manifolds M.

4. Let M be a manifold. Consider the map $\wedge : H^k(M) \times H^l(M) \to H^{k+l}(M)$, $([\omega], [\eta]) \mapsto [\omega \wedge \eta]$. Prove that $\wedge$ is well-defined on the level of cohomology. (Then, since $[1] \wedge [\omega] = [\omega]$, and $[\alpha] \wedge [\beta] = (-1)^{\deg(\alpha)\deg(\beta)}[\beta] \wedge [\alpha]$, it follows that the vector space $H^\bullet(M) := \oplus_{i=0}^{\dim M} H^i(M)$, equipped with the operations $+$ and $\wedge$, has the structure of a *(graded) algebra*.

5. Let S^2 denote the unit sphere in $\mathbb{R}^3$, $\{(r_1, r_2, r_3) | r_1^2 + r_2^2 + r_3^2 = 1\}$. We saw early on that S^2 admits the atlas $\mathcal{A} = \{(U_i^\pm, \pi_i^\pm\}_{i=1,2,3}$ where

$$U_i^+ = \{r_i > 0\} \cap S^2, \; U_i^- = \{r_i < 0\} \cap S^2$$

and π_i^+ and π_i^- are both projection away from the ith coordinate, e.g., $\pi_1^\pm(r_1, r_2, r_3) = (r_2, r_3)$.

   (a) Is $\mathcal{A}$ a Euclidean oriented atlas?

   (b) Let

   $$\sigma = \frac{r_1 dr_2 \wedge dr_3 - r_2 dr_1 \wedge dr_3 + r_3 dr_1 \wedge dr_2}{(r_1^2 + r_2^2 + r_3^2)^{3/2}}$$

   be a two-form on $\mathbb{R}^3 \backslash \{0\}$. Prove that σ restricted to S^2 is closed.

   (c) Prove that σ restricted to S^2 is not exact (Hint: evaluate $\int_{S^2} \sigma$. You should not need to get stuck using partitions of unity, though you should justify how you can avoid them. Remember part (a)).