Math 535a Homework 8 (half weight)

Due Friday, April 28, 2017 by 5 pm

Please remember to write down your name on your assignment.

1. *Transversality.* In this exercise, we will show that "transversality is suitably generic," in the sense of Sard's theorem, as mentioned in class.

   (a) Let F and V be manifolds, and M another manifold with $A \subset M$ a submanifold. Let $\Phi : F \times V \to M$ be a smooth map which is transverse to A. Show that there is a dense subset of $v \in V$ such that $\phi_v := \Phi(-, v) : F \to M$ is transverse to A. *Hint: consider the projection $\Phi^{-1}(A) \hookrightarrow A \times V \to V$, and show that if $v \in V$ is a regular value of this map, then ϕ_v is transverse to A.*

   (b) Let K and L be submanifolds of $\mathbb{R}^N$. Show that there are arbitrarily small translations of K that are transverse to L. That is, show that for any $\epsilon > 0$, there exists a $v \in \mathbb{R}^N$ with $||v|| < \epsilon$, such that $T_v(K)$ is transverse to L, where $T_v : \mathbb{R}^N \xrightarrow{\sim} \mathbb{R}^N$ is the diffeomorphism sending x to $x + v$. (hint: it may be helpful to use part (a).)

   (c) Show that if $E \to M$ is a vector bundle, and $s \in \Gamma(E)$ a smooth section, then $M_s = im(s)$ is *isotopic* to M_0 in E. Show that if $M_s \subset V_\epsilon(E)$ for some orthogonal structure on E, then M_s remains isotopic to M_0 as submanifolds of $V_\epsilon(E)$. (in other words, show that the isotopy constructed would stay in $V_\epsilon(E)$.

   (*Note*: Recall that a pair of submanifolds L_0, $L_1 \subset M$ are *isotopic* if there is a manifold L and a smooth homotopy $\phi_t : L \to M$, $t \in [0, 1]$ such that (a) each ϕ_t is an embedding, and (b) $\text{im}(\phi_0) = L_0$, $\text{im}(\phi_1) = L_1$.)

   (d) (*double weight sub-problem*) Finally, let K and L be submanifolds of M (assume K, L, and M are compact for simplicity). Recall that the tubular neighborhood theorem implies that there is a diffeomorphism $\Phi_K : V_K \cong U_K$, where U_K is an open neighborhood of K in M and V_K is an open neighborhood of the zero section $\underline{K}$ in the normal bundle $\nu(K) = (TM|_K)/(TK)$ of the form of an λ-disk bundle $V_\lambda = \{(q, w) \in \nu(K) | ||w||_q < \lambda\}$ for some orthogonal structure on $\nu(K)$.

   Prove that there is an arbitrarily close isotopy of K, $\tilde{K}$, with $\tilde{K}$ is transverse to L. *Note*: *Arbitrarily close* means that one can find such a $\tilde{K}$ with image in any open neighborhood U of K. Since any open neighborhood U contains the image of a subdisk bundle of a tubular neighborhood,[1] it will suffice to show that one can pick $\tilde{K}$ of the form $\tilde{K} = im(\Phi_K \circ s)$, where V_K is an open subset of the zero section in $\nu(K)$ with $\Phi_K : V_K \xrightarrow{\cong} U_K \subset U \subset M$ a tubular neighborhood of K with image contained

[1] The fact that any open neighborhood U of a compact submanifold K possesses a tubular neighborhood of K, is a short consequence of the tubular neighborhood theorem applied to the submanifold K of U, or a compactness argument applied to to show that any tubular neighborhood of L in M contains a sub-neighborhood of uniform size which stays in U.

in U, and $s \in \Gamma(\nu(K))$ is a section whose image lies in V_K. Note that by part (c), since $im(s)$ and $im(0)$ are isotopic in V_K, it follows that $\Phi_K(im(s))$ and $\Phi_K(im(0))$ are isotopic in M.

Detailed hint:

- By Whitney embedding, it suffices to assume $M \subset \mathbb{R}^N$.

- Define a map
$$f : K \times \mathbb{R}^N \to \mathbb{R}^N$$
sending (k, x) to $k + proj_k(x)$, where $proj_k : \mathbb{R}^N \to (T_kK)^\perp \hookrightarrow \mathbb{R}^N$ is the orthogonal projection onto the subspace $(T_kK)^\perp$, the orthogonal complement to T_kK in $T_kM \subset T_k\mathbb{R}^N$. Observe by construction that f factors as
$$K \times \mathbb{R}^N \overset{proj}{\to} (TK)^\perp \overset{\Phi}{\to} \mathbb{R}^N, \tag{0.1}$$
where the first map, $proj : K \times \mathbb{R}^N \to (TK)^\perp$, a morphism of vector bundles, is given by fibrewise orthogonal projection $(k, v) \mapsto (k, proj_k(v))$, and the second map Φ sends (p, w) to $p + w$.

- Argue that for small δ, f maps $K \times B_\delta(0)$ to a predetermined tubular neighborhood U_M of M in $\mathbb{R}^N$. Meaning, U_M is a given neighborhood of M in $\mathbb{R}^N$ for which there exists a neighborhood V_M of the zero section of M in $\nu(M) \cong (TM)^\perp$ and a diffeomorphism $\Phi_M : V_M \overset{\cong}{\to} U_M$ sending the zero section identically to M). In particular, for such U_M the tubular neighborhood induces a map $\bar{\pi} := \pi \circ \Phi_M^{-1} : U_M \overset{\cong}{\to} V_M \overset{\pi}{\to} M$. Composing with the map $\bar{\pi}$, one obtains a map $F : K \times B_\delta(0) \to M$.

- Verify that, after shrinking δ, it's possible to ensure that F always has image in the specified neighborhood U of K.

- Apply part (a) to conclude that, after possibly shrinking δ further, for generic $v \in B_\delta(0)$, $K_v := F(K \times \{v\})$ is transverse to L. (To apply part (a), one needs to verify that F is transverse to L. Why is this true? *Hint*: shrink δ enough so that the map $\bar{\pi} \circ \Phi_K : V_\delta(TK^\perp) \to M$ gives a tubular neighborhood of K in M, where $V_\delta(\nu(K))$ denotes the δ-disc bundle of $TK^\perp$, using the orthogonal structure from $\mathbb{R}^N$. Now apply the fact that $F : K \times B_\delta(0) \to M$ factors as
$$K \times B_\delta(0) \overset{proj}{\to} V_\delta(TK^\perp) \overset{\Phi_K}{\to} M, \tag{0.2}$$
Why does this help show that F is transverse to L?)

- Deduce from (0.2) that for any $v \in B_\delta(0)$, $F(K \times \{v\}) = \tilde{K}_s = \Phi_K(s(K))$ for some section $s \in \Gamma(\nu(K))$ with image in $V_\delta(\nu(K))$. Conclude the result.

2. Let $M \subset \mathbb{R}^{n+1}$ be a compact connected oriented submanifold of Euclidean space, without boundary. You may assume the generalization of the *Jordan curve theorem*: $\mathbb{R}^{n+1} \backslash M$ has

two connected components, one of which is bounded and one of which is unbounded.

For each point $x \in \mathbb{R}^{n+1} \backslash M$, define

$$\sigma_x : M \to S^n$$
$$p \mapsto (p - x)/||p - x||.$$

Prove that if x and y are in the same component of the same component of $\mathbb{R}^{n+1} \backslash M$, then σ_x is smoothly homotopic to σ_y. Prove that x is in the bounded component if and only if $\deg(\sigma_x) = \pm 1$, and x is in the unbounded component if and only if $\sigma_x \simeq \textit{constant}$. (*Hint*: for the first, consider an x with coordinate x_{n+1} greater than the maximum x_{n+1} acheieved on M. For the second, consider a point x just below a point on M with maximal x_{n+1} value).

3. (a) Let Z be a compact submanifold of Y, both oriented, with $\dim Z = \frac{1}{2} \dim Y$. Prove that $Z \bullet Z = (Z \times Z) \bullet \Delta$, where $\Delta = \{(x, x) | x \in Y\} \subset Y \times Y$ is the diagonal of Y.

   Remark: Note that in cases when K is not transverse to L (such as $K = L$), we defined $K \bullet L := K \bullet \tilde{L}$ (or $\tilde{K} \bullet L$), where $\tilde{L}$ is a small isotopy of L as in Problem 1. By our discussion in class, this intersection number is independent of such choice of small isotopy.

   (b) Let $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with its canonical orientation (induced by the one on $\mathbb{R}^2$, $K = \{(s, 0) | s \in \mathbb{R}/\mathbb{Z}\}$, and $L = \{(t, nt) | t \in \mathbb{R}/\mathbb{Z}\}$, both equipped with the orientation induced by the one on $\mathbb{R}/\mathbb{Z}$. Calculate, with proof, the intersection number $K \bullet L$.