Math 535A Lecture 1

Monday, January 9, 2017

Review of topological spaces

Recall the following definition from point-set topology:

Definition 1. *A* topological space *is a pair* $(X, \mathcal{T})$ *(often abbreviated* X*, with* $\mathcal{T}$ *left implicit) where* X *is a set and* $\mathcal{T}$ *is a family of subsets of* X *such that*

1. *The empty set* $\emptyset$ *and* X *are both elements of* $\mathcal{T}$*,*
2. *Any finite intersection of elements of* $\mathcal{T}$ *is in* $\mathcal{T}$*, and*
3. *arbitrary unions of elements of* $\mathcal{T}$ *are again contained in* $\mathcal{T}$ *(to recall notation, we state this precisely as: "if* $\{U_\alpha\}_{\alpha \in I}$ *is a collection of subsets of* X *with* $U_\alpha \in \mathcal{T}$ *for all* $\alpha \in I$*, then* $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$*. ")*

The collection of subsets $\mathcal{T}$ is called a *topology* on X, and the subsets $U \in T$ are called the *open sets* of X.

Example 1. Any set S has at least two topologies:

- The *trivial topology* $\mathcal{T} = \{\emptyset, S\}$; and
- The *discrete topology* $\mathcal{T} = \mathcal{P}(S) = \{\text{all subsets of } S\}$.

In particular, because there is no set of all sets, there is no set of all topological spaces.

Example 2. Let $X := (X, d)$ be a *metric space*, which we recall is a set X equipped with a function $d : X \times X \to [0, \infty)$ (called the *metric* or *distance function*), satisfying

- $d(x, y) = 0$ if and only if (iff) $x = y$; and
- $d(x, y) = d(y, x)$ for all $x, y \in X$; and
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Given a point $x \in X$ and any real number $\epsilon > 0$, recall that the *open ball of radius* ϵ *centered at* x is

$$B_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\}.$$

From a metric space $X := (X, d)$, we obtain a topology $\mathcal{T}$ on X (called the *metric topology*) as follows: we say a set $U \subset X$ is open, e.g., in $\mathcal{T}$, if at any $x \in U$, there is a $B_\epsilon(x)$ contained in U for some ϵ. One might also call this the topology "generated" by the sets $B_\epsilon(x)$.

Recall that we use the notation $\mathbb{R}^n := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies}}$ for n-dimensional Euclidean space. $\mathbb{R}^n$ can be equipped with the structure of a metric space, using the (usual) Euclidean metric

$$d(x, y) = ||x - y|| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2};$$

in particular $\mathbb{R}^n$ can be thought of as a topological space.

Recall some further definitions from point set topology:

- A topological space X is *Hausdorff* if for any two points $x, y \in X$, there exists a neighborhood[1] U of x and V of y with $U \cup V = \emptyset$ (so U and V "separate" x from y).
- A subset $A \subset X$ is *dense* if every non-empty open set in X contains an element of A.
- A topological space X is *separable* if it contains a countable dense subset. (For instance $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense and countable, hence $\mathbb{R}^n$ is separable).
- A topological space $X := (X, \mathcal{T})$ is *second countable* if there is a countable subset $\mathcal{T}_0 \subset \mathcal{T}$ which generates the topology $\mathcal{T}$, in the sense that elements of $\mathcal{T}$ are (potentially arbitrary) unions of elements of the countable collection $\mathcal{T}_0$. It is not hard to see that if X is a metric space which is separable, then X (thought of as a topological space) is second countable: let if $A \subset X$ denotes the countable dense set, take $\mathcal{T}_0$ be the collection of balls of rational radius centered at the points of A.

Let's recall also some methods of constructing topological spaces

1. If X and Y are topological spaces, then the Cartesian product $X \times Y$ is again a topological space whose open sets are generated by products of open sets in X with that in Y.
2. Suppose $Y := (Y, \mathcal{T})$ is a topological space, X a set, and $i : X \hookrightarrow Y$ an injective map (for instance, X could be a subset of Y and i could just be the inclusion). Then, X carries an *induced* (or *subspace*; though that notation is usually reserved for when X is actually a subset) topology as follows: we say $U \subset X$ is open if and only if it's of the form $i^{-1}(U_Y)$ for U_Y an open subset of Y.[2] If X was a subset of Y, this is equivalent to: $U \subset X$ is open if and only if it's of the form $U_Y \cap X$, where $U_Y \subset Y$ is open.

   Example 3. Let $X = S^1 = \{(x, y) | x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Then, X inherits the structure of a topological space from the topology on $\mathbb{R}^2$ discussed above.
3. Suppose now $X := (X, \mathcal{T})$ is a topological space, Y is a set, and $\rho : X \twoheadrightarrow Y$ a surjective map of sets. Then, Y inherits a topology, called the *quotient topology*, from X and ρ, as follows: we say $U \subset Y$ is open if and only if the preimage $\rho^{-1}(U)$ is open in X.

   Example 4. The closed interval $[0, 1] \subset \mathbb{R}$ is a topological space (equipped with the subspace topology from the inclusion $[0, 1] \hookrightarrow \mathbb{R}$), and there is a surjective map $\rho : [0, 1] \to S^1$ given by identifying 0 and 1 (concretely, one could realize this as the map $t \mapsto (\cos 2\pi t, \sin 2\pi t)$). The topological space $[0, 1]$ and the map ρ give S^1 the structure of a topological space.

Continuous functions and homeomorphisms

In Examples 3 and 4, we equipped S^1 with two different topologies; let us call the resulting topological spaces S^1_a and S^1_b. We'd like to compare them, and say they're the same, for instance. Along the way, it would be helpful to recall the general method by which we compare topological spaces: continuous maps.

[1] For the purposes of our class, a *neighborhood* of a point p refers to an "open neighborhood" of the point p, i.e., an open set containing p.

[2] Recall the notation for *preimage* of a subset: if $g : A \to B$ is a map of sets, and $S \subset B$ a subset, then the preimage of S is $g^{-1}(S) := \{x \in A | g(x) \in S\}$.

Definition 2. *A map $f : X \to Y$ between topological spaces*[3] *is said to be* continuous *if, for every open subset $U \subset Y$, the preimage $f^{-1}(U)$ is open in X.*

For a metric space, the above definition is equivalent to the more familiar definition expressible in terms of ϵs and δs.

Recall, that *inverse* of a map of sets $f : A \to B$, if it exists, is the unique map $g : B \to A$ satisfying $g \circ f = id_A$, $f \circ g = id_B$. In light of the two conditions which need to be satisfied, sometimes g is referred to as a two-sided inverse, and maps $g : B \to A$ which satisfy just one of the conditions $g \circ f = id_A$ or $f \circ g = id_B$ are called one-sided (or left or right respectively) inverses. A necessary and sufficient condition for an inverse to exist is for f to be *bijective*, e.g., injective and surjective. For a topological space, we

Definition 3. *A* continous map *between topological spaces $f : X \to Y$ is said to be a* homeomorphism *if it has an (two-sided) inverse $g : Y \to X$ which is also continuous. Equivalently, f is a continuous bijection with continuous inverse.*

If $f : X \to Y$ is a homeomorphism, then $U \subset X$ is open if and only if $f(U) \subset Y$ is open. Therefore, we can use homeomorphisms to faithfully "translate" properties about the topology of X to that of Y and back. We might say "homeomorphisms" are the "isomorphisms" in the "category" of topological spaces (more on this point of view next time).

Example 5. Continuing the discussion in examples 3 and 4 and at the top of this section, it turns out that there is a homeomorphism $S^1_a \cong S^1_b$, so we can think of the two constructions of the topology on S^1 as producing the "same" topological space.

[3] meaning, a map of the underlying sets