Math 535a Homework 3

Due Monday, February 12, 2018 by 5 pm

Please remember to write down your name on your assignment.

1. (*Half-weight*) Let $x_0 \in \mathbb{R}^n$ be a point and $r_1 < r_2$ positive real numbers. Construct (with proof) a C^∞ function $h : \mathbb{R}^n \to \mathbb{R}$ which equals 1 inside the ball of radius r_1 around x_0 and which equals 0 outside the ball of radius r_2 around x_0. Such functions are collectively called *smooth bump functions*.

   (*note*: you may wish to first construct the corresponding function in one dimension $f : \mathbb{R} \to \mathbb{R}$ first, and then use this one-dimensional construction as a stepping stone to the general case).

2. The *immersion version of the implicit function theorem* is stated as follows: Suppose $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are open sets, and $f : U \to V$ is a smooth map which is an *immersion*, meaning the rank of df_p is m (or equivalently df_p is *injective*) for every $p \in U$. Then, for any $p \in U$, there exists a possibly smaller open set $\tilde{U} \subset U$ still containing p, a smaller open set $\tilde{V} \subset V$ containing $f(\tilde{U})$, and a diffeomorphism $G : \tilde{V} \to Z \overset{\text{open}}{\subset} \mathbb{R}^n$, where Z is some open set in $\mathbb{R}^n$, such that $G \circ f : \tilde{U} \to Z \subset \mathbb{R}^n$ agrees with the "model immersion"

$$\begin{aligned} \pi_{imm} &: \mathbb{R}^m \to \mathbb{R}^n \\ (x_1, \dots, x_m) &\mapsto (x_1, \dots, x_m, \underbrace{0, \dots, 0}_{n-m \text{ times}}). \end{aligned}$$

   Prove the immersion version of the implicit function theorem, assuming only the inverse function theorem. *Hint*: the proof is very similar to that of the submersion version of the implicit function theorem.

3. Show that the two definitions of a *submanifold* $Y^m \subset N^n$ given in class are equivalent. Here are the two definitions again:
   - *Definition 1*: A subset $Y \subset N^n$ is a submanifold of dimension m if it is the image of some *embedding*[1] $M^m \hookrightarrow N^n$.
   - *Definition 2*: A subset Y of a manifold N^n is said to be a submanifold of dimension m if at at every point $p \in Y$, there exists a chart (U, ϕ) in N's maximal atlas, containing (and centered at) p, such that $\phi(U \cap Y) = \phi(U) \cap \{x_{m+1} = x_{m+2} = \cdots = x_n = 0\} = \phi(U) \cap (\mathbb{R}^m \times \{0\})$.

   Note: typically we consider "proper submanifolds," which are subsets $Y \subset N^n$ as in either of the two above definitions, such that the inclusion $Y^m \to N^n$ is a *proper* map. This "proper" prefix can be appended onto either definition. Note that if Y is compact, it is automatically a proper subset of any manifold N^n. Also, it is a fact (that we have not proved currently) that a submanifold $Y \subset N$ is proper if and only if Y is closed in N.

[1] An *embedding* $M^m \to N^n$ is a C^∞ map f which is (a) an immersion, and (b) injective (one-to-one) as a map of sets.

4. Prove the following result: if $f : M^m \to N^n$ is a submersion between two smooth manifolds, or more generally if f is simply a smooth map and $y \in N$ is a regular value of f, then $S := f^{-1}(y)$ has the structure of a smooth submanifold of M of dimension $m - n$.

   Remarks:
   - You are welcome to use (and probably should use) the implicit function theorem.
   - The result you are being asked to prove is slightly stronger than the result stated as a "Corollary" in class. Namely, you are being asked to prove not just that S can be given the structure of a smooth manifold, but in fact that S with the smooth manifold structure that it can be given is naturally a *submanifold* of N, in the sense of the homework problem below. You may use either definition of *submanifold*, as you will prove below that both definitions are equivalent.
   - **Hint**: In class, we sketched the construction of a chart containing any point $p \in S$. You are welcome to recall, with detail, this construction, and may want to analyze it to produce a chart (U, ϕ) of a neighborhood of p in M, whose intersection with S maps to the intersection of $\phi(U)$ with $\mathbb{R}^m \times \{0\}$. Finally, you must prove that transition functions between any two such charts are C^∞).

5. Prove that $S^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ can be given the structure of an n-dimensional manifold by exhibiting it as the regular value of some smooth map between manifolds.

6. Recall that $M_n(\mathbb{R})$ denotes the set of $n \times n$ matrices with $\mathbb{R}$-entries; we showed this a manifold of dimension n^2. Let $O(n) = \{A \in M_n(\mathbb{R}) | AA^T = I\}$ be the *orthogonal group*, where A^T is the *transpose* of A. Consider the map

$$\begin{aligned} \phi : M_n(\mathbb{R}) &\to \mathrm{Sym}(n) \\ A &\mapsto AA^T \end{aligned}$$

   where $Sym(n) = \{B \in M_n(\mathbb{R}) | B = B^T\}$ is the set of *symmetric matrices.*

   (a) Show that $\mathrm{Sym}(n)$ is a submanifold of $M_n(\mathbb{R})$ (and in particular a manifold), and compute its dimension. (**Hint**: It may be helpful to first prove, then apply, the following general Lemma: If V is a finite-dimensional vector space, it canonically has the structure of a smooth manifold (meaning in a manner independent of choice of basis of V); furthermore, if $W \subset V$ is a linear subspace, then W is naturally a submanifold of V. It may be helpful to pick a basis on V to get a chart, but if you do this, then show that the resulting differentiable structure is independent of choice of basis.).

   (b) Prove that $I \in \mathrm{Sym}(n)$ is a regular value of ϕ.

   (c) Prove that $O(n)$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension?

   (d) Prove that $O(n)$ is compact.

7. *Categories and functors*. Earlier in class, we defined the notion of a *category*[2] $\mathcal{C}$; examples given include *topological spaces* **Top**, and *vector spaces over a field* k, $\mathbf{Vect}_k$.

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ from category $\mathcal{C}$ to $\mathcal{D}$ is an assignment, to every object of $\mathcal{C}$, an object of $\mathcal{D}$, and an induced map on morphism spaces. More precisely, a *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is specified by the following data:

- A map on object $F : \text{ob } \mathcal{C} \to \text{ob } \mathcal{D}$
- For every pair of objects X, Y, a map on morphism spaces $F = F_{XY} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$, which satisfies:
  - F sends identity morphisms to identity morphisms (so $F(id_X) = id_{F(X)}$, where $X \in \text{ob } \mathcal{C}$.), and
  - F is compatible with compositions, in the sense that $F(g) \circ F(f) = F(g \circ f)$ for any objects X, Y, Z and morphisms $g \in \hom(Y, Z)$, $f \in \hom(X, Y)$.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$, written as

$$G : \mathcal{C}^{op} \to \mathcal{D},$$

consists of the following data: [3]

- A map on object $G : \text{ob } \mathcal{C} \to \text{ob } \mathcal{D}$
- For every pair of objects X, Y, a map on morphism spaces $G = G_{XY} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(G(Y), G(X))$ (note the order reversal), which satisfies:
  - G sends identity morphisms to identity morphisms (so $G(id_X) = id_{G(X)}$, where $X \in \text{ob } \mathcal{C}$.), and
  - G is compatible with compositions, in the sense that $G(f) \circ G(g) = G(g \circ f)$ for any objects X, Y, Z and morphisms $g \in \hom(Y, Z)$, $f \in \hom(X, Y)$.

In other words, a contravariant functor is specified by the same sort of data as a covariant functor, except the order of morphisms in the target is reversed in passing from the source to the target category.

(a) *The category of algebras* Let $\mathbf{Alg}_k$ be the category of k-algebras, defined as follows:
   - the *objects* of $\mathbf{Alg}_k$ are k-algebras, [4]
   - The morphisms in $\mathbf{Alg}_k$ from A to B are the set of *k-algebra homomorphisms.* [5]
   - For any triple A, B, C, the composition map $\hom_{\mathbf{Alg}_k}(B, C) \times \hom_{\mathbf{Alg}_k}(A, B) \to \hom_{\mathbf{Alg}_k}(A, C)$ is simply composition of homomorphisms.

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[2]Recall that a *category* $\mathcal{C}$ consists of objects, morphisms, and a composition rule satisfying two conditions: (a) for each object X, there exists an identity morphism $1_X \in \hom_{\mathcal{C}}(X, X)$ satisfying (a) $1_X \circ f = f$, $g \circ 1_X = g$ for any objects Y, Z and morphisms $f \in \hom_{\mathcal{C}}(Z, X)$ and any $g \in \hom_{\mathcal{C}}(X, Y)$, and (b) composition is associative, meaning that for any triple f, g, and h which are composable $h \circ (g \circ f) = (h \circ g) \circ f$. (here composable means that $h \in \hom_{\mathcal{C}}(Z, W)$, $g \in \hom_{\mathcal{C}}(Y, Z)$ and $f \in \hom_{\mathcal{C}}(X, Y)$ for some objects X, Y, Z, W)

[3]A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is the same as a covariant functor from the *opposite category* $\mathcal{C}^{op}$ of $\mathcal{C}$ to $\mathcal{D}$, hence the notation. We will not elaborate on this point more here.

[4]Let k be any field. For our purposes, a *k-algebra* A is a vector space over k equipped with a multiplication map $\times : A \times A \to A$ which is a *bilinear map.* This map, in addition to being bilinear, should satisfy two extra properties: the mutiplication map (a) is associative, and (b) there is a multiplicative identity $1 \in A$ satisfying $1 \cdot \alpha = \alpha \cdot 1 = \alpha$, for all $\alpha \in A$. (elsewhere, such A are frequently called *associative unital algebras*

[5]For our purposes, an *k-algebra homomorphism* $F : A \to B$ is a linear map of vector spaces which is compatible with the multiplication maps, meaning that $F(\alpha \cdot \beta) = F(\alpha) \cdot F(\beta)$. F should also preserve the identity elements, so $F(1) = 1$; this is frequently elsewhere called a *unital algebra homomorphism.*

Prove that $\mathbf{Alg}_k$ indeed a category.

(b) For any topological space X, prove that $C^0(X) = \{\text{continuous functions } f : X \to \mathbb{R}\}$ is an $\mathbb{R}$-algebra, with multiplication given by multiplication of functions: $f \times g$ is the function whose value at p is $f(p) \cdot g(p)$.

(c) Prove that there is a contravariant functor $F = C^0(-) : \mathbf{Top}^{op} \to \mathbf{Alg}_{\mathbb{R}}$ defined as follows:

- on the level of objects, given an object $X \in \text{ob } \mathbf{Top}$, i.e., a topological space, $F(X)$ associates the object $C^0(X) \in \text{ob } \mathbf{Alg}_{\mathbb{R}}$.

- on the level of morphisms, given a morphism (i.e., a continuous map) $f \in \hom_{\mathbf{Top}}(X, Y)$, $F(f)$ associates the pullback map $f^* : C^0(Y) \to C^0(X)$ (i.e., $F(f) = f^* \in \hom_{\mathbf{Alg}_{\mathbb{R}}}(F(Y), F(X)) = \hom_{\mathbf{Alg}_{\mathbb{R}}}(C^0(Y), C^0(X))$, where $f^*(h) = h \circ f$. You should verify that f^* is indeed a morphism of algebras (so far in class, we only proved it was a morphism of sets!).

(d) Let **Set** denote the category of sets; objects are sets and morphisms are maps of sets (you may assume this is a category). Prove that there is a functor, the *forgetful functor*

$$Forget : \mathbf{Alg}_{\mathbb{R}} \to \mathbf{Set}$$

which on objects, associates to an algebra the underlying set (forgetting any algebra structure), and on morphisms, associates to any algebra homomorphism the same map, now thought of as a map of sets. Show that this functor is faithful but *not* "full"[6] (*Hint*: to show not full, find a pair of objects X, Y in $\mathbf{Alg}_{\mathbb{R}}$, such that there is a map of sets from X to Y which is not an $\mathbb{R}$-algebra homomorphism).

Note: The *composition* of functors $Forget \circ C^0(-)$ is the functor we discussed in class, which associates to any space X the *set* $C^0(X)$.

[6]A functor is *faithful*, respectively *full* if for any two objects X, Y in $\mathcal{C}$, the map on morphisms $F : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{C}}(FX, FY)$ is injective, respectively surjective.