Math 535a Homework 4

Due Monday, February 26, 2018 by 5 pm

Please remember to write down your name on your assignment.

1. Let $M \subset \mathbb{R}^N$ be a submanifold of $\mathbb{R}^N$, and $p \in M$ a point. Verify that the two extrinsic definitions of tangent space to M at p (the second one requiring us to further assume that M is $f^{-1}(y)$ for some smooth function $f : \mathbb{R}^N \to \mathbb{R}^{N-m}$ and some regular value $y \in \mathbb{R}^{N-m}$) and the first intrinsic definition of tangent space given in class are all naturally isomorphic.
   Hints: To go from extrinsic definition 1 to extrinsic definition 2, first show that if α is any curve in $\mathbb{R}^N$ through p with image in M, then $df_p(\alpha'(0)) = 0$; this creates an inclusion in one direction.

2. Let $M = \{(x, y, z) \in \mathbb{R}^3 | z = \sqrt{x^2 + y^2}\}$.
   (a) Show that $M - \{(0, 0, 0)\}$ is a 2-dimensional submanifold of $\mathbb{R}^3 - \{(0, 0, 0)\}$.

   (b) Let $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^3$ be a smooth curve with image contained in M, such that $\alpha(0) = (0, 0, 0)$. Show that $\alpha'(0) = (0, 0, 0)$. *Possible hint*: Write $\alpha(t) = (x(t), y(t), z(t))$, note that $z(t)^2 = x(t)^2 + y(t)^2$, and first prove that $z'(0) = 0$.

   (c) Use part (b) to show that M is not a submanifold of $\mathbb{R}^3$. *Hint*: otherwise, what would the tangent space $T_{(0,0,0)}M$ be?

3. Given a manifold M and a point p, as defined in class, let C_p^M denote the collection of all parametrized curves in M passing through p at 0:

   $$C_p^M := \{(I, \alpha) | I \text{ any interval containing } 0, \alpha : I \to M \text{ smooth with } \alpha(0) = p\}.$$

   As in class, we defined an equivalence relation $\sim$ on C_p^M as follows: pick any chart (U, ϕ) in M's atlas containing p, we say that $(I, \alpha) \sim (J, \beta)$ if $(\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0)$[1]
   You may assume that this is an equivalence relation.
   (a) Verify that $\sim$ is moreover independent of choice of chart (U, ϕ) in M's maximal atlas containing p.

   Once this is done, we can define the *tangent space to p at M* by $T_pM := C_p/\sim$.

   (b) If W is an open subset of $\mathbb{R}^m$, and $q \in W$ any point, verify that there is an isomorphism of sets

   $$C_q^W/\sim \xrightarrow{\sim} \mathbb{R}^m$$

[1]Note that while the image of α respectively β, may not lie in U, we can always find a subinterval around 0 $\bar{I} \subset I$ respectively $\bar{J} \subset J$ such that $\alpha(\bar{I})$ respectively $\beta(\bar{J})$ is contained in U (why?). Hence $\phi \circ \alpha$ and $\phi \circ \beta$ are defined in a small neighborhood around 0 in I respectively J; moreoever the derivative at 0 is independent of which small subinterval we choose.

which sends an equivalence class $[(I,\alpha)]$ to $\alpha'(0)$, for any chosen representative (I,α) in the equivalence class (why is this well-defined?)

(c) Prove that there exists a unique vector space structure on T_pM such that for each chart (U,ϕ) containing p, the map

$$\Phi : T_pM \to C^{\phi(U)}_{\phi(p)}/\sim \xrightarrow{\sim} \mathbb{R}^m$$

is a linear isomorphism.

4. Give a detailed proof of the equivalence between the three definitions of T_pM given in class. Then, prove that the construction of the derivative

$$df_p : T_pM \to T_{f(p)}N$$

is the same for the three definitions, meaning the following: If $T_p^{(i)}M$ denotes the ith construction of the tangent space, for $i = 1, 2, 3$, and

$$df_p^{(i)} : T_p^{(i)}M \to T_{f(p)}^{(i)}N$$

the corresponding three different constructions of the derivative, then show that for any M and p and any i, j there are isomorphisms

$$g^{(ij)}_{p,M} : T_p^{(i)}M \cong T_p^{(j)}M$$

which intertwine the derivative maps, in the sense that $df_p^{(i)} = g^{(ji)}_{f(p),N} \circ df_p^{(j)} \circ g^{(ij)}_{p,M}$ (where $g^{(ji)}_{p,M} = (g^{(ij)}_{p,M})^{-1}$).

5. Let Γ be a group and M a smooth manifold. A (C^∞) *action* of Γ on M is a group homorphism ρ from Γ to the group $\mathrm{Diff}(M)$ of diffeomorphisms on M. If $\gamma \in \Gamma$ and $x \in M$, we write $\gamma x = \rho(\gamma)(x)$ for the image of x under the diffeomorphism $\rho(\gamma)$.

Recall from class that the *quotient space* M/Γ of the action Γ on M is the set of equivalence classes of the equivalence relation $\sim$ defined by $x \sim y$ iff $y = \gamma x$ for some $\gamma \in \Gamma$.

(a) We say the action of Γ on M is *discontinuous* if, for every compact subset K of M, the set $\{\gamma \in \Gamma | K \cap \gamma K \neq \emptyset\}$ is finite. We say the action of Γ on M is *free* if $\gamma x \neq x$ for every $x \in M$ and $\gamma \in \Gamma - \{\mathrm{id}\}$.

Prove that if Γ acts freely and discontinously on M, then the quotient M/Γ naturally has the structure of a smooth manifold.

(b) Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ act on $S^n \subset \mathbb{R}^{n+1}$ by sending $x \mapsto -x$. Using the standard manifold structure on S^n (either as given above via expressing S^n as a preimage or as studied on homework last week), prove that $S^n/\mathbb{Z}_2$ has the structure of a manifold, which is diffeomorphic to $\mathbb{R}P^n$, equipped with the smooth manifold structure which you defined on your homework last week: (with charts $U_i = \{x_i \neq 0\}, \phi_i : U_i \mapsto \mathbb{R}^n$, $[x_0 :$

$$\cdots x_n] \mapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i}\right).$$