Math 535a Homework 5

Due Friday, March 9, 2018 by 5 pm

Please remember to write down your name on your assignment.

1. Let $M = f^{-1}(y)$ be the preimage of a regular value $y \in \mathbb{R}^{N-m}$ of a smooth function $f : \mathbb{R}^N \to \mathbb{R}^{N-m}$. (for instance, $M = S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3 = f^{-1}(1)$, where $f : (x, y, z) \mapsto x^2 + y^2 + z^2$).

   (a) Let $\widetilde{TM} = \{(x, v) \in \mathbb{R}^N \times \mathbb{R}^N | x \in M, v \in \ker df_x\}$. Show that as defined, $\widetilde{TM}$ is a smooth submanifold of $\mathbb{R}^N \times \mathbb{R}^N$ of dimension $2m$ (where M is an m-dimensional manifold).

   (b) Prove that there is a diffeomorphism between $\widetilde{TM}$ and the *tangent bundle of M* as defined in class:

   $$\widetilde{TM} \cong TM$$

   in a manner compatible with projection to M; meaning that, if $\tilde{\pi} : \widetilde{TM} \to M$ is the map sending $(x, v) \mapsto v$, then there is a commutative diagram

   $$\begin{array}{ccc} \widetilde{TM} & \xrightarrow{\cong} & TM \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{=} & M \end{array}$$

   (It follows that, for instance, $TS^2 \cong \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 | x \in S^2 | v \cdot x = 0\}$.

2. Let M^m be a manifold of dimension m and $p \in M$ a point. Recall that $\mathcal{F}_p \subset C^\infty(p)$ is the ideal of germs of functions on M which vanish at $p \in M$. Let $\mathcal{F}_p^k$ be the ideal of $C^\infty(p)$ generated by $f_1 \cdots f_k$, where $f_i \in \mathcal{F}_p$. (This means that every element of $\mathcal{F}_p^k$ is a sum $\sum_i g_i f_{1i} \cdots f_{ki}$, where $g^i \in C^\infty(p)$, and $f_{ij} \in \mathcal{F}_p$).

   (a) Prove that, in every set of local coordinates $(x_1, \ldots, x_k)$ around the point p, an element $f \in \mathcal{F}_p^k$ has a Taylor expansion which vanishes to order k. You may assume a version of Taylor's approximation theorem stated in class.

   (b) Compute the dimension of $\mathcal{F}_p^k / \mathcal{F}_p^{k+1}$.

   (c) Construct a smooth manifold along with a map to M, $E \xrightarrow{\pi} M$ whose "fiber" $E_p = \pi^{-1}(p)$ at the point $p \in M$ is $\mathcal{F}_p^1 / \mathcal{F}_p^3$.

3. Let $f : M \to N$ be a smooth map between manifolds. Prove that the following diagram commutes:

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{f^*} & \Omega^0(M) \\ \downarrow d & & \downarrow d \\ \Omega^1(N) & \xrightarrow{f^*} & \Omega^1(M) \end{array}$$

4. Give a detailed proof that the cotangent bundle T^*M is a smooth manifold and that the projection map $\pi : T^*M \to M$ is a smooth map.

5. Let f and g be smooth real-valued functions on a manifold M. Prove that $d(fg) = fdg + gdf$.

6. Let $i : S^1 = [0, 2\pi]/(0 \sim 2\pi) \to \mathbb{R}^2$ be the map $\theta \mapsto (\cos(\theta), \sin(\theta))$. Compute $i^*((x^2 + y)dx + (3 + xy^2)dy)$.[1]

[1]As discussed in class, the notation $f_1dx + f_2dy$, where f_1 and f_2 are smooth functions on $\mathbb{R}^2$, is a common shorthand for the 1-form $\mathbb{R}^2 \to T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ sending $\vec{x}$ to $(\vec{x}, (f_1(\vec{x})dx + f_2(\vec{x})dy))$.