Math 535a Homework 7 (1.5x weight)

Due Friday, April 27, 2017 by 5 pm

Please remember to write down your name on your assignment.

1. *Orientability* Prove that real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd. (**Hint**: Show that the antipodal map on the n-sphere S^n (with a fixed orientation of your choice) is orientation preserving if and only if n is odd. Why does this help?).

2. Let S^2 denote the unit sphere in $\mathbb{R}^3$, $\{(r_1, r_2, r_3) | r_1^2 + r_2^2 + r_3^2 = 1\}$. We saw early on that S^2 admits the atlas $\mathcal{A} = \{(U_i^{\pm}, \pi_i^{\pm}\}_{i=1,2,3}$ where
$$U_i^+ = \{r_i > 0\} \cap S^2,\ U_i^- = \{r_i < 0\} \cap S^2$$
and π_i^+ and π_i^- are both projection away from the ith coordinate, e.g., $\pi_1^{\pm}(r_1, r_2, r_3) = (r_2, r_3)$.

   (a) Is $\mathcal{A}$ a Euclidean oriented atlas?

   (b) Let
$$\sigma = \frac{r_1 dr_2 \wedge dr_3 - r_2 dr_1 \wedge dr_3 + r_3 dr_1 \wedge dr_2}{(r_1^2 + r_2^2 + r_3^2)^{3/2}}$$
be a two-form on $\mathbb{R}^3 \backslash \{0\}$. Prove that σ restricted to S^2 is closed.

   (c) Prove that σ restricted to S^2 is not exact (Hint: evaluate $\int_{S^2} \sigma$. You should not need to get stuck using partitions of unity in the computation, though you should justify why you can avoid them (given that they appear in the definition of the integral! Also, remember part (a)).

3. Suppose that $M = M_1 \coprod M_2$. Then prove that
$$H^k_{dR}(M) = H^k_{dR}(M_1) \oplus H^k_{dR}(M_2).$$

4. Use the Mayer-Vietoris sequence to prove that
$$H^k_{dR}(S^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}.$$
You may assume, as input, the computation of the de Rham cohomology of $\mathbb{R}^n$ and S^1. Inductively prove from there that
$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

5. Give a careful computation of the de Rham cohomology (and hence, the Euler characteristic) of a genus g surface Σ_g, using the sketch given in class or another method of your choice. You may feel free to use your favorite presentation of Σ_g, including using a picture. (the "careful" part refers to the cohomological computations)

6. *Computing degrees in simple cases.*
   (a) Let $S^1 = \mathbb{R}/\mathbb{Z}$, and fix an orientation on S^1 of your choice. For every $k \in \mathbb{Z}$, there is a map from S^1 to itself.
   $$\begin{aligned} F_k : S^1 &\to S^1 \\ [z] &\mapsto [kz]. \end{aligned}$$
   You may assume this map is smooth and well-defined. Compute the degree of this map, with respect to your choice of orientation (you should use the same orientation for both sides).
   (b) Equipping S^n with whichever orientation you please, compute the degree of the reflection map $(x_1, \ldots, x_{n+1}) \mapsto (-x_1, x_2, \ldots, x_{n+1}) : S^n \to S^n$.

7. *A problem about degrees.* Let $M^n \subset \mathbb{R}^{n+1}$ be a compact connected oriented n-dimensional submanifold of $n+1$-dimensional Euclidean space, without boundary. You may assume the generalization of the *Jordan curve theorem*: $\mathbb{R}^{n+1} \backslash M^n$ has two connected components, one of which is bounded and one of which is unbounded.

   For each point $x \in \mathbb{R}^{n+1} \backslash M^n$, define
   $$\begin{aligned} \sigma_x : M^n &\to S^n \\ p &\mapsto (p - x)/||p - x||. \end{aligned}$$
   Prove that if x and y are in the same component of the same component of $\mathbb{R}^{n+1} \backslash M^n$, then σ_x is smoothly homotopic to σ_y. Prove that x is in the bounded component if and only if $\deg(\sigma_x) = \pm 1$, and x is in the unbounded component if and only if $\sigma_x \simeq$ *constant*. (*Hint*: for the first, consider an x with coordinate x_{n+1} greater than the maximum of x_{n+1} which is achieved on M. For the second, consider a point x just below a point on M with maximal x_{n+1} value).