Math 535b Final Project: Lecture and Paper

1. Overview and timeline

The goal of the Math 535b final project is to explore some advanced aspect of Riemannian, symplectic, and/or Kähler geometry (most likely chosen from a list below, but you are welcome to select and plan out a different topic, with instructor approval). You will then give two forms of exposition of this topic:

- A 50 minute lecture, delivered to the class.
- A 5+ page paper exposition of the topic. Your paper must be typeset (and I strongly recommend LaTeX)

The timeline for these assignments is:

- Your lectures will be scheduled during the last two weeks of class, April 15-19 and 22-26 - both during regular class time and during our make-up lecture slot, Wednesday between 4:30 and 6:30pm. We will schedule precise times later this week. If you have any scheduling constraints, please let me know ASAP.
- Your final assignment will be due the last day of class, April 26. If it is necessary, this can be extended somewhat, but you will need to e-mail me in advance to schedule a final (hard) deadline.

Some general requirements and/or suggestions:

(1) *In your paper, you are required to make reference to, in some non-trivial fashion, at least one original research paper.* (you could also mention any insights you might have learned in your lecture though this is not necessary. Although many of the topics will be "textbook topics," i.e., they are now covered as advanced chapters in textbooks, there will be some component of the topic for which one or more original research papers are still the "best" or "most definitive" references. Reading research papers is a crucial skill for further graduate research, so the goal here is to give you some experience with original references.
(2) Exposition is important, as are key definitions, intuition, and some key proofs or proof sketches. It is *not* necessary to give rigorous arguments of every result.
(3) The lecture and paper probably should not be identical in emphasis, as the formats are very different. In particular, your goal for the lecture is to convey the relevant ideas at the level your fellow students will understand in lecture. Your target audience for the paper is again your fellow students, but the aim is for your document to be more like a section of a chapter or an expository introductory article.

2. Lecture Requirements

It is *strongly recommended* you give a practice lecture to some subset of the students in the class (or any other graduate students who might be in attendance).

3. Paper Requirements

Write a 5+ page paper (single spaced, font at most 12 pt, margins at most 1.5 inch). Please cite all references used (you are welcome to cite "unofficial references," but never as a primary source; i.e., all results cited should have at least one original reference. By original reference we mean book or published article; arXiv preprints are permissible though not recommended).

4. Meetings

Each student should schedule one or more meetings with me to (a) go over the material, and (b) finalize an outline for your paper and/or lecture (possibly separate meetings).

5. List of possible topics

Here are some topics grouped by category, as well as some references (which you are not required to use; you may select your own references but let's make sure to discuss them in advance).

5.1. Constructions of symplectic and Kähler manifolds and submanifolds. This is a large class of topics, including:

- *Hamiltonian group actions* and *Hamiltonian reduction* as a method for producing new manifolds. Symplectic (and in fact Kähler) *toric* manifolds are a class of manifolds with Hamiltonian torus actions that (a) can be described as Hamiltonian reductions, and (b) admit a beautiful classification in terms of polytopes. You could discuss this class of manifolds (which includes projective space and is closed under products), give examples, and state and/or sketch the relevant existence and classification theorems.

  References include Cannas da Silva's book and notes on toric manifolds (the latter is available here: `https://people.math.ethz.ch/~acannas/Papers/toric.pdf`), Audin's "The topology of torus actions on symplectic manifolds," ...
- *Symplectic fibrations* (as covered in McDuff-Salamon's "Introduction to symplectic topology" Chapter 6) and more generally *Lefschetz fibrations*. The goal is to talk about how these can be used to both construct symplectic manifolds, and in the latter case of Lefschetz fibrations, construct new Lagrangian submanifolds ("vanishing cycles") and symplectomorphisms which are not Hamiltonian isotopic to the identity ("Dehn twists.")
- The symplectic sum of symplectic manifolds along a symplectic submanifold, after Gompf. References include Gompf's original paper "A new construction of symplectic manifolds" and McDuff-Salamon's Introduction to Symplectic Topology, Chapter 7.1.
- Symplectic and Kähler blowups: references include McDuff-Salamon's Introduction to Symplectic Topology, Chapter 7.2,
- Constructing symplectic structures on open manifolds via h-principle methods.

5.2. Constructing submanifolds and/or embeddings. This broad area naturally has some overlap with the topic above. Some possibilities include

- *Constructing symplectic submanifolds, after Donaldson.* Any Kähler manifold which is *projective*, i.e., can be realized as a submanifold of $\mathbb{P}^n$ has many complex submanifolds, for instance obtained by intersecting with a complex hyperplane. A remarkable result of Donaldson states that the same is true for a symplectic manifold which need not be Kähler. In Donaldson's paper "Symplectic submanifolds and almost-complex geoemtry" (see Corollary 6 for the most general statement and Theorem 1 for the main result which can be used to prove it). Give an exposition of Donaldson's proof; Auroux's simplification of it in the paper "A remark about Donaldson's construction of symplectic submanifolds" may also be useful.
- *The Kodaira embedding theorem.* We will probably at least mention this in class, references include Wells, Voisin.
- *Constructing Lagrangian submanifolds*: You could survey one or more known methods for constructing Lagrangian submanifolds including: constructing Lagrangian spheres via Lefschetz fibrations, (look at the discussion on symplectic fibrations above), constructing Lagrangian submanifolds via generating functions, the Polterovich Lagrangian surgery of Lagrangian submanifolds.
- Gromov's h-principle and applications (to embedding problems and existence problems for symplectic manifolds and Lagrangian/isotropic embeddings and/or immersions).
- Embedding problems for symplectic domains, including *Gromov's non-squeezing result and other embedding problems.* There are remarkable obstructions (going beyond just volume) to embedding certain standard symplectic domains (i.e., manifolds-with-boundary) such as balls into others (such as polydisks or cylinders), a story that begins with *Gromov's non-squeezing theorem.* Discuss this theorem and sketch a proof, and then one modern variant (to be discussed with me). For instance, McDuff-Schlenk's paper "The embedding capacity of 4-dimensional symplectic ellipsoids" exhibits rather rich embeding behavior.

5.3. Classification and geography questions.

- Symplectic non-Kähler manifolds - you could write specifically about a few constructions of symplectic non-Kähler manifolds, by Thurston, McDuff, and Gompf. Thurston, McDuff, Gompf.

5.4. Riemannian geometry and topology.

- *Characteristic classes from curvature.* Characteristic classes are certain cohomology classes associated to vector bundles satisfying a list of axioms. Much like cohomology or homology itself, there are many definitions of these classes which obey the axioms; each definition gives different insight into what these classes "are." The idea here is to define (using curvature of connections, i.e., rudimentary *Chern-Weil theory*) the higher Chern classes of a complex vector bundle E on a manifold M. You might then show well-definedness (independence of choice of connection), and prove that the resulting cohomology classes satisfy the axioms of Chern classes. If F is a real rank $2k$ vector bundle on M without a complex structure, you could also define a class called the *Euler class* of F; check that this class agrees with the top Chern class of F with respect to any complex structure on F.

Finally, you may wish to state and/or sketch a proof of a generalized version of the Gauss-Bonnet theorem, known as the *Gauss-Bonnet-Chern theorem*: it shows that if you take the Euler class of TM and integrate it over M, you get the Euler characteristic.

References include: Chern's "Global differential geometry" chapter 1, Wells Chapter 3, Auroux's course notes, Chern's original proof of the generalized Gauss-Bonnet theorem, ...

- Theorems in Riemannian geometry: there are beautiful results in Riemannian geometry that use curvature and connections to deduce insteresting global geometric or topological properties. As a sample, one has
  - The sphere theorem says that if a manifold admits a metric with curvature staying strictly in a certain interval (in a precise quantifiable sense), then the manifold is a sphere.
  - Gromov's Betti number bound gives an upper bound for the Betti numbers of a compact Riemannian manifold in terms of its diameter and its curvature.
- Arnold's nearby Lagrangian conjecture. You could give a survey of this and discuss some special cases. References TBD.

5.5. Miscellaneous.

- Thom's work on cobordism. References include Hirsch's "Differential topology."
- Morse theory is a powerful tool for analyzing manifolds via looking at sublevel sets of a particularly nice function (with "non-degenerate critical points"), and seeing how they change (in hopefully simple ways) as one crosses critical values (the sublevel sets change by "attaching handles"). Your goal is to define the notion of a *Morse function* (following, say, Milnor's book on Morse theory or Hirsch's "Differential topology" chapter 6), some basic results about them (existence/genericity, the Morse Lemma on their local form near a critical point, etc.) and then develop the theory from scratch in order to prove one or more facts about manifolds:
  - the fact that any compact manifold is homotopy equivalent to a finite CW complex (as defined in say Math 540 or equivalent).
  - The Morse inequalities, giving lower bounds on the number of critical points of a function.
  - Morse homology (only sketchily) and its equivalence with ordinary homology.