Math 641 Homework 6: Spectral sequences

Due Thursday, May 6, 2021 by 5 pm

Please remember to write down your name and ID number. We will refer to pages/sections from Hatcher's *Spectral sequences in algebraic topology* by [HatcherSS] and McCleary's *User's guide to spectral sequences* by [McCleary].

1. Let $s \neq 1$ be a fixed integer. Give an example (with proof) of a chain complex and a filtration on it so that in the associated spectral sequence, $\partial_r = 0$ except when $r = s$, and $\partial_s \neq 0$.

2. *Fibrations and Euler characteristic.* Prove, using a method of your choice covered in class, that for any fibration $F \to E \to B$ (assuming B and F are path connected; and you may also assume if it helps for spectral sequence arguments — though it is not necessary to do so — that the local coefficient system $\{H_*(F_x)\}_{x \in B}$ is trivial), the Euler characteristic is multiplicative $\chi(E) = \chi(B) \cdot \chi(F)$.

3. *Proving Leray-Hirsch using the Leray-Serre spectral sequence*: Solve problems 2 and 3 in Section 1.2, p. 51 of [HatcherSS], which outline a method to prove the Leray-Hirsch theorem using spectral sequences.

4. Use the Leray-Serre spectral sequence to compute $H_*(\Omega(S^3 \vee S^3))$.

5. Show using the Leray-Serre spectral sequence that an even dimensional sphere cannot be the total space of a spherical fibration over a sphere (regardless of dimension of fiber and base). (hint: first show that the base of such a fibration must be simply connected, then study the Serre spectral sequuence).

6. *[McCleary] part of Exercise 5.13.* The symplectic groups, $Sp(n)$, are the analogues of the special orthogonal or unitary groups over the quaternions. There are fibrations defined analogously, $Sp(n-1) \to Sp(n) \to S^{4n-1}$ (define these fibrations). From $Sp(1) = S^3$, compute the cohomology ring $H^*(Sp(n))$ for all n.

7. *[McCleary] Exercise 6.11.* Show that the transgression $\tau : H^{n-1}(S^{n-1}) \to H^n(S^n)$ for the fibration $S^{n-1} \to S^*(T^*S^n) \to S^n$, the sphere bundle of the tangent bundle of S^n, is given by the Euler number of S^n.