Math 641, Spring 2021 – "Topics in topology" (this semester advanced algebraic topology, continuation of Math 540).

Instructor: Sheel Ganatra, sheel.ganatra@usc.edu

Schedule: Most weeks, MW 9:30am – 11am.

today: F 9:30–11

next week: WF 9:30–11 (no class Monday)
(1/18-1/22)

after that: MW, occasional F (makeup times).

Grading:
- 50% HW assignments
  - assigned every week or two, mostly optional
  - each assignment, choose 1 problem to submit for a grade (& grade is for completion/effort).
- 50% final paper. (5-10 pages about a topic from a list of choices or another topic w/ instructor approval)

Zoom Policy: please have camera on during class...

Overview of course: This is a second semester course in algebraic topology. (following Math 540, and to some extent, Math 535a), covering:

(1) cohomology theory

(2) Poincaré duality for manifolds (and fundamental classes)

(3) vector bundles

(4) characteristic classes of vector bundles

Along the way, time permitting, we may say some things about:

- fibrations + spectral sequences,
- classifying spaces,
- applications to manifolds (e.g., to embeddings, immersions, cobordisms).
- the cobordism ring (+ generalized homology theories).

In some more detail:

In first semester alg. topology, we start to learn about homotopy invariants of top. spaces, i.e., invariants up to homotopy equivalence (e.g., fundamental group, homology, cohomology, higher homotopy groups), & methods of computing them, & their structure on spaces of interest.

What types of spaces? To start, we're interested in:

- $\mathbb{R}^k$, D^k, S^{k-1}, etc.

- spaces that admit a decomposition into simple pieces.
  - simplicial complexes, or more generally

  - CW complexes

    (recall a CW complex is any space X constructed inductively by

level 0 $\qquad X^0 := \bigcup$ points

$\vdots$

level n $\qquad$ given maps $\{\varphi_\alpha : \underset{= \partial D^n_\alpha}{S^{n-1}_\alpha} \to X^{n-1}\}_\alpha$, we form X^n by attaching each D^n_α to X^{n-1} along φ_α

$$X^n = X^{n-1} \amalg \coprod_\alpha D^n_\alpha \Big/ x \in \partial D^n_\alpha \sim \varphi_\alpha(x).$$

either stops at some level N & $X = X^N$

or $\quad X = \bigcup X^n$.

- manifolds:

  - A (top.) manifold of dimension k is a (Hausdorff, second countable) space X which is locally homeomorphic to $\mathbb{R}^k$.

  - A (smooth) manifold is a top. manifold equipped with a smooth atlas
$$\{(U_\alpha, \phi_\alpha) \mid \underset{\text{open}}{U_\alpha \subset M}, \ \phi_\alpha : U_\alpha \xrightarrow[\text{homeo}]{\cong} \phi_\alpha(U_\alpha) \underset{\text{open}}{\subset} \mathbb{R}^k\}$$
s.t. $\underbrace{\{U_\alpha\} \text{ cover } M}_{\text{atlas}}$, & $\forall\, \alpha, \beta$, $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$

are C^∞ maps between open subsets of $\mathbb{R}^k$.

smooth manifolds have a notion of differentiable functions, & differentiable maps

Manifolds often admit CW/simplicial structures, e.g.,

- any compact manifold is at least homotopy equiv. to a CW complex (Appendix of Hatcher)
- any smooth compact manifold admits a CW complex structure (up to homeomorphism)
- any top. compact manifold admits a CW structure if $\dim M < 4$ or $\dim M > 4$ (unknown if always does when $\dim M = 4$??)

Rmk: not every top. manifold (e.g., in dim 4, but in other dimensions too) admits a smooth structure.

Questions:
- when are spaces / manifolds equivalent or not?
  - homotopy equiv.?
  - homeomorphic?
  - diffeomorphic?
- when does $M \hookrightarrow M'$ (embedding of manifold)
- when is X (top space) $\simeq$ (htpy equiv) ~~[illegible]~~ a manifold?
- when is $M = \partial W$?
- more generally, when are M_0, M_1 cobordant ($\partial W = M_0 \sqcup M_1$)

(a manifold-with-boundary is locally homeomorphic to open subsets of $\mathbb{H}^n = \{\vec{x} \in \mathbb{R}^n \mid x_1 \geq 0\}$).

$\mathbb{H}^2$.)

- 

W $\quad \partial W = S^1$.

- 

$\partial W = (S^1 \cup S^1) \sqcup S^1$.

so $S^1 \cup S^1$ and S^1 are cobordant, & W is a cobordism between them.

what types of invariants will we use to address these questions?

Previously, introduced

singular homology:

singular chains $\quad C_k(X) = \bigoplus_{\sigma : \Delta^k \to X} \mathbb{Z}\langle\sigma\rangle \quad$ w/ $\partial_{(k)} : C_k(X) \to C_{k-1}(X)$

$\sigma : \Delta^k \to X$ "singular simplices"

$$\partial\sigma := \sum (-1)^i \sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_k]}$$

"restriction of σ to $\partial\Delta^k = \coprod_{\text{faces}} \Delta^{k-1}$"

Had $\partial_{(k-1)} \circ \partial_k = 0$ (chain complex), and

$$H_k(X) := H_k(\{C_k(X), \partial_k\}_k) = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}.$$

covariantly functorial: $f: X \to Y$ induces $f_\#: C_\bullet(X) \to C_\bullet(Y)$ &

$$f_*: H_\bullet(X) \to H_\bullet(Y).$$

& (local: excision/Mayer-Vietoris); • Have $H_\bullet(X,A) := H_k\left(C_\bullet(X,A) := C_\bullet(X)/C_\bullet(A)\right)$ (for $A \subset X$)

• $H_k(X;R)$ & $H_k(X,A;R)$ via $C_k(X;R) := C_k(X) \otimes_{\mathbb{Z}} R = \bigoplus_{\sigma: \Delta^k \to X} R\langle \sigma \rangle$.

We'll start by defining singular cohomology, a dual theory:

$$\underset{\text{singular cochains}}{C^k(X;R)} := \operatorname{Hom}_{\mathbb{Z}}(\overset{\text{singular chains}}{C_k(X)}; R).$$

inherits $\delta_k = \partial^*_{k+1}: C^k(X;R) \to C^{k+1}(X;R)$

i.e., $\delta_{(k)}(f) := f \circ \partial_{(k+1)}$.

& similarly $\delta_{k+1} \circ \delta_k = 0$; so can define $H^k(X;R) := \frac{\ker \delta_k}{\operatorname{im} \delta_{k-1}}$.

(as before, have $H^k((X,A);R)$, Mayer-Vietoris/Excision, etc.,

contravariant functoriality ($f: X \to Y$ induces $f^*: H^\bullet(Y) \to H^\bullet(X)$).

Initially/a priori, this has same information as $H_k(X)$, packaged differently.

But, it turns out $H^\bullet(X) := \bigoplus H^k(X)$ has more structure, a ring (cup product), c.f. wedge product

(ex: $H^k(S^2 \vee S^4) \cong H^k(\mathbb{CP}^2)$, but ring structures are different so $S^2 \vee S^4 \not\simeq \mathbb{CP}^2$)

Product comes from $\Delta: X \to X \times X$ diagonal embedding, which induces

$$\begin{tikzcd} H^\bullet(X)\otimes H^\bullet(X) \arrow[r, "\cong"] \arrow[rr, bend right, "\text{cup product}"'] & H^\bullet(X\times X) \arrow[r, "\Delta^*"] & H^\bullet(X) \end{tikzcd}$$

(can alternatively think of Δ as inducing a coproduct on $H_\bullet(X)$:

$$\begin{tikzcd} H_\bullet(X) \arrow[r] \arrow[dr, "\Delta_*"'] & H_\bullet(X)\otimes H_\bullet(X) \\ & H_\bullet(X\times X) \arrow[u] \end{tikzcd}$$
)

<u>Poincaré duality</u>: tells us e.g. that $H^k(M) \cong H_{n-k}(M)$ (M^n cpct <u>oriented</u> manifold) $\cong H_k(M)^\vee$ mod torsion / over a field, & e.g. that

$$\begin{tikzcd} H^k(M)/\text{tors} \otimes H^{n-k}(M)/\text{tors} \arrow[r, "\text{cup product}"] & H^n(M) \arrow[r, "\cong"] & \mathbb{Z} \end{tikzcd}$$

P.D. "$\int_M$"

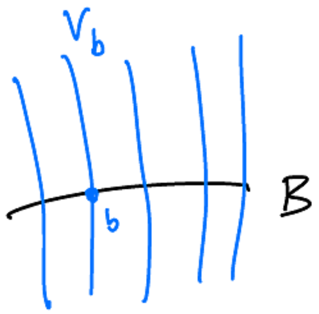
is "unimodular" $\Rightarrow$ perfect pairing over any field.

(cor: on M^{4k}, $H^{2k}(M)$ inherits a symm. bilinear (unimodular) pairing $H^{2k}\times H^{2k} \to \mathbb{Z}$, called the <u>intersection form</u>.)

<u>Next major topic</u> (another impetus for cohomology): vector bundles.

Roughly, a vec. bundle is a "topologized" family of vector spaces $\{V_b\}_{b\in B}$ over some base B.

(can make precise by saying $\exists$ a total space E & map $\pi: E \to B$ w/ $\pi^{-1}(b)$ are vector spaces, satisfying "local triviality" condition)



We're often interested in studying such objects up to isomorphism

e.g., can ask is a given E trivial? (means $E \cong B\times\mathbb{R}^k$)

(Mobius bundle over S^1 is <u>not</u> trivial).

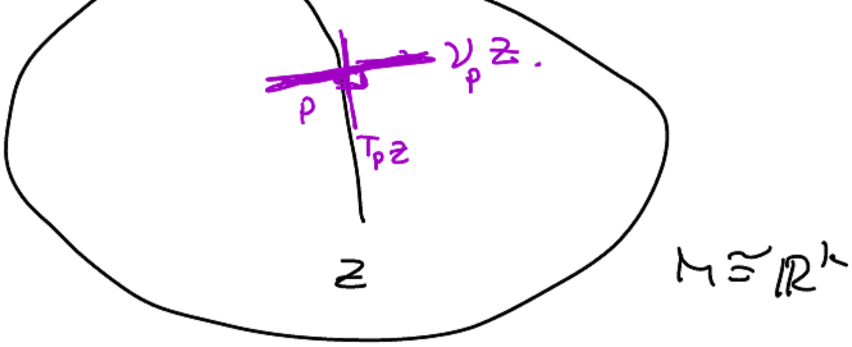
[why? one reason comes from manifold theory:

- any smooth M has a tangent bundle $TM \to M$. w/ fibers tangent spaces

- any $Z \hookrightarrow M$ submanifold has a <u>normal bundle</u>

$\nu_{Z\subset M}$

$\downarrow$
Z



(& more generally, have fiber bundles, fibrations (weaker), principal bundles (very related to vector bundles))

How to classify such bundles?

<u>Characteristic classes</u> are an important tool:

A char. class associates to $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ vec bundle an element $c(E) \in H^\bullet(B)$ (w/ some coefficients)

satisfying some axioms (e.g., $c(E) = 0$ if E is trivial, functoriality, ---)

We'll introduce & study several families of such char. classes:

- Stiefel-Whitney classes (live in $H^\bullet(B; \mathbb{Z}/2)$ $w_i(E) \in H^i(B; \mathbb{Z}/2)$)
- Chern classes (defined for <u>complex</u> vector bundles, get $c_i(E) \in H^{2i}(B; \mathbb{Z})$).
- <u>Pontryagin</u> classes (defined for any E, $p_i(E) \in H^{4i}(B; \mathbb{Z})$)
  (defined from c_i by roughly $p_i(E) := c_{2i}(E \otimes \mathbb{C})$)

& applications to topology.

For instance, get $\overset{\text{numerical}}{\text{invariants}}$ of $\overset{\text{a priori}}{\text{(smooth)}}$ compact manifolds:

For any collection of characteristic classes $d_1, \dots, d_k$ (not nec. distinct), so that $d_1(TM) \cup \dots \cup d_k(TM) \in H^n(M)$, if dimension of M.
M is oriented, $H^n(M) \xrightarrow{\cong} \mathbb{Z}$ (or a field like $\mathbb{Z}/2$ in Stiefel-Whitney case)

(not nec. in Stiefel-Whitney case). $\int_M$

& so get a number, called <u>characteristic numbers</u> of M. (e.g., $w_1^n \in \mathbb{Z}/2$, $w_1^{n-2} w_2 \in \mathbb{Z}/2$).

Sample theorem (which we hopefully will try to talk about; otherwise great paper topic!):

<u>Thm</u>: M_0, M_1 are (unoriented) cobordant <u>iff</u> all Stiefel-Whitney numbers coincide.
(Thom) In particular, $M^n = \partial W^{n+1}$ iff all Stiefel-Whitney numbers of M are 0.

time permitting, we'll take a digression into <u>spectral sequences</u>.

(disconnected case $H^n(M) \xrightarrow[\langle [M], - \rangle]{(\text{not } \cong)} \mathbb{Z}/2$; e.g., M connected, then $M \amalg M$ is nullbordant, b/c it's $\partial(M \times [0,1])$. Indeed all Stiefel-Whitney #s are $2 \cdot$ (Stiefel-Whitney #s of M) $= 0$).