Universal coefficient theorems for homology and cohomology

Last time: X top. space $\underset{\text{last sem.}}{\rightsquigarrow}$ $C_\bullet(X)$ singular homology chain complex (w/ $\mathbb{Z}$-coefficients).

$$C_\bullet(X) = \{ \to C_n(X) \xrightarrow{\partial_{(n)}} C_{n-1}(X) \xrightarrow{\partial_{(n-1)}} \cdots \to C_0(X) \to 0 \}$$

boundary operators

chain complex: $\partial_{(n-1)} \circ \partial_{(n)} = 0$ $\left(\rightsquigarrow \text{singular homology } H_n(X) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} \right)$.

From this, (rather than immediately take homology) we can form the ith singular co-chain group w/ G-coefficients (G any abelian group):

$$C^i(X;G) := \operatorname{Hom}_{\mathbb{Z}}(C_i(X), G)$$

(could take $G = \mathbb{Z}$, or something else),

"Hom of Abelian groups = $\mathbb{Z}$-modules"

and the singular co-chain complex w/ G-coeffs.

$$C^\bullet(X;G) = \{ \cdots \xleftarrow{\delta} C^n(X;G) \xleftarrow{\delta} C^{n-1}(X;G) \leftarrow \cdots \xleftarrow{\delta} C^0(X;G) \leftarrow 0 \}$$

$= \operatorname{Hom}_{\mathbb{Z}}(C_\bullet(X), G)$ where $\delta_i = \partial^*_{i+1} = (-)\circ \partial_{i+1} : C^i(X;G) \to C^{i+1}(X;G)$ for all i.

Again we have $\delta \circ \delta = 0$ (called a co-chain complex),

added b/c differential increases, rather than lowers degree

and can take $H^i(X;G) := \dfrac{\ker \delta^i : C^i(X;G) \to C^{i+1}(X;G)}{\operatorname{im} \delta^{i-1} : C^{i-1}(X;G) \to C^i(X;G)}$.

Rmks: • For a pair (X,A) can similarly define $C^\bullet(X,A;G)$ by taking $\operatorname{Hom}_{\mathbb{Z}}(C_\bullet(X,A); G)$.

• $C_i(X) := \bigoplus_{\sigma: \Delta^i \to X} \mathbb{Z}\langle\sigma\rangle$, i.e., $C_i(X) = \operatorname{Free}(\operatorname{Sing}^i(X))$

set of all singular simplices $\sigma: \Delta^i \to X$.

So $C^i(X;G) = \operatorname{Hom}_{\mathbb{Z}}(C_i(X), G)$

$= \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Free}(\operatorname{Sing}^i(X)), G)$

$= \operatorname{Maps}_{\text{Set}}(\operatorname{Sing}^i(X), G)$.

e.g., $C^0(X;G) = \operatorname{Maps}_{\text{Set}}(\operatorname{Sing}^0(X), G)$

$= \operatorname{Maps}_{\text{set}}(X, G)$ = functions from $X^{\text{discrete}} \to G$.

$\uparrow$
X as a set (forgotten all topology).

- Any map $f: X \to Y$ induces $f_\#: C_\bullet(X) \to C_\bullet(Y)$ chain map, hence induces $f^\# = (f_\#)^* = (-)\circ f_\# : C^\bullet(Y;G) \to C^\bullet(X;G)$ a co-chain map (meaning again $f^\# \circ \delta_Y = \delta_X \circ f^\#$), hence a map $f^* = [f^\#]: H^\bullet(Y;G) \to H^\bullet(X;G)$, i.e., $H^i(-;G)$ is <u>contravariantly</u> functorial. (as opposed to covariantly like H_i).

(similarly for $f: (X,A) \to (Y,B)$, get f^* between cohomologies in <u>opposite</u> direction)

In light of the fact that $C^\bullet(X;G)$ are determined as $\mathrm{Hom}_{\mathbb{Z}}(C_\bullet(X), G)$, we might ask:

<u>Q</u>: what's the relationship between $H^\bullet(X;G)$ and $H_\bullet(X)$?

To put things on level footing, let's recall we can also take singular chains w/ G-coeffs: (G any ab. group)

$C_n(X;G) := C_n(X) \otimes_{\mathbb{Z}} G$, w/ induced ∂ $(:= \partial \otimes \mathrm{id}_G)$, and

$\rightsquigarrow H_\bullet(X;G)$

implicitly $H_n(X;\mathbb{Z})$

<u>Q</u>: what's the relationship between $H_n(X)$ and $H_n(X;G)$?

More generally, can consider any chain complex $C_\bullet = \{ \to C_n \xrightarrow{\partial} \cdots \to C_0 \to 0\}$;

Q: what's the relation between $\{H_n(C_\bullet)\}_n$ and

- $\{H^n(\mathrm{Hom}_{\mathbb{Z}}(C_\bullet, G))\}_n$?
- $\{H_n(C_\bullet \otimes_{\mathbb{Z}} G)\}_n$?

<u>Example</u>:

$$\begin{array}{ccccccccc} & & \text{deg } 2 & & \text{deg } 1 & & \text{deg } 0 & & \\ C = & & 0 & \to & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[\partial_1]{[2,0]} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

has homology

$H_1 = \ker \partial_1 \cong \underline{\mathbb{Z}}$ ($= 0 \oplus \mathbb{Z}$)

$H_0 = \mathrm{coker}\, \partial_1 \cong \underline{\mathbb{Z}/2\mathbb{Z}}$.

$$\mathrm{Hom}(C,\mathbb{Z}): \quad 0 \leftarrow \mathbb{Z}\oplus\mathbb{Z} \xleftarrow[\delta_0]{\begin{bmatrix}2\\0\end{bmatrix}} \mathbb{Z} \leftarrow 0$$

not linear duals of each other!!

$$\left\{ \mathrm{Hom}(\mathbb{Z}^2,\mathbb{Z}) \cong \mathbb{Z}^2 \right\}$$

$$H^1 = \operatorname{coker} \delta_0 = \frac{\mathbb{Z}\oplus\mathbb{Z}}{2\mathbb{Z}\oplus 0} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

$$H^0 = \ker \delta_0 = 0$$

Roughly speaking, it seems free part of H_i contributes to H^i,
torsion part of H_{i-1} contributes to H^i.

$$C \otimes_{\mathbb{Z}} \mathbb{Z}/2 : \quad 0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{[0\ 0]} \mathbb{Z}/2 \to 0$$

$$\rightsquigarrow H_i(C_\bullet; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i=1 \\ \mathbb{Z}/2 & i=0 \end{cases} \qquad (\underline{\text{not}}\ H_i(C_\bullet)\otimes_{\mathbb{Z}} \mathbb{Z}/2 \,!)$$

__Exercise:__ look at $\mathrm{Hom}(C, \mathbb{Z}/2)$ to see similar discrepancies

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General $C_\bullet$:

Note there is a natural map

$$H^n = H^n(\mathrm{Hom}(C_\bullet, G)) \ni [f] \xrightarrow{\beta} \mathrm{Hom}_{\mathbb{Z}}(H_n(C_\bullet), G)$$

given by

$$\beta([f])([c]) := f(c).$$

(exercise: this is independent of choice of representatives of $[f]$ and $[c]$, i.e., note that

$$(f+\delta g)(c) = f(c) + \delta g(c) = f(c) + g(\partial c) \qquad (\text{b/c } \delta g = g\circ\delta)$$
$$= f(c) \quad (\text{b/c } \partial c = 0).)$$

There's also a natural map

$$H_n(C_\bullet)\otimes G \xrightarrow{\alpha} H_n(C_\bullet \otimes G)$$

$$\alpha([x]\otimes g) := [x\otimes g]$$

More refined question: how to measure failure of β, α to be isomorphisms?

note $C_\bullet(X)$ is free.

__Theorem:__ (universal coefficient theorem for cohomology) (UCT) for any __free__ chain complex $C_\bullet$ (means each C_i is free),

there is a (functorial) __natural__ in $C_\bullet$ and G SES for each n:

the map β from above

$$0 \to \mathrm{Ext}(H_{n-1}(C_\bullet), G) \to H^n(\mathrm{Hom}(C_\bullet, G)) \xrightarrow{\beta} \mathrm{Hom}_{\mathbb{Z}}(H_n(C_\bullet), G) \to 0$$

new term! Sometimes called $\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}, G)$ (but we often leave $\mathbb{Z}$, 1 implicit; can leave 1 implicit b/c $\mathrm{Ext}^k_{\mathbb{Z}}(A,B)=0$ for $k>1$).

Furthermore, this sequence <u>splits</u> (naturally in G, but <u>not</u> <u>naturally</u> in $C_\bullet$).

Recall: A SES $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ splits if $\exists$ $k: C \to B$ w/ $j \circ k = id_C$

If k exists, it need not be unique, and k induces $A \oplus C \xrightarrow[\cong]{(i,k)} B$, so get $B \cong A \oplus C$.

non-split ex:

$$0 \to \mathbb{Z} \xrightarrow{i = \times 2} \mathbb{Z} \xrightarrow{j = \text{projection}} \mathbb{Z}/2 \to 0.$$

So UCT (+ a choice of splitting) gives

$$H^n(\mathrm{Hom}_{\mathbb{Z}}(C_\bullet, G)) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(C_\bullet), G) \oplus \mathrm{Ext}(H_{n-1}(C_\bullet), G).$$

but this isomorphism is not <u>natural</u> in $C_\bullet$; i.e., a chain map $f: C_\bullet \to C_\bullet'$ induces a map of SES's above (& in particular on cohomology), but not necessarily respecting the direct sum decompositions for any choice of splitting. (<u>exercise</u>)

<u>Thm</u> (UCT for homology): For a free chain complex $C_\bullet$, $\exists$ an exact sequence

$$0 \to H_n(C_\bullet) \otimes_{\mathbb{Z}} G \xrightarrow{\alpha} H_n(C_\bullet \otimes G) \to \mathrm{Tor}(H_{n-1}, G) \to 0$$

sometimes called $\mathrm{Tor}_1^{\mathbb{Z}}$ (but can suppress $\mathbb{Z}$ if implicit & 1 again b/c $\mathrm{Tor}^{\mathbb{Z}}_{>1} \equiv 0$).

natural (functorial) in $C_\bullet$ (w/r/t. chain maps) and G. The sequence splits (naturally in G, but <u>not</u> in $C_\bullet$).

The next goal is to define Ext/Tor, then we'll see how to prove UCTs.

For any R-modules M, N, can define $\mathrm{Ext}^i_R(M,N)$ and $\mathrm{Tor}_i^R(M,N)$, with

commutative ring (only so we don't have to worry about left vs. right modules; R associative is otherwise sufficient).

$$\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N), \quad \mathrm{Tor}_0^R(M,N) = M \otimes_R N;$$

with $\mathrm{Ext}^i_R(M,N)$ (resp. $\mathrm{Tor}_i^R(M,N)$) for $i>0$ measuring "the failure at M" of $\mathrm{Hom}_R(-,N)$ (resp. $(-)\otimes_R N$) to be 'exact.'"

A functor f is <u>exact</u> if whenever have SES $0 \to A \to B \to C \to 0$ then

- if f contravariant,

$0 \to f(C) \to f(B) \to f(A) \to 0$ is exact.

- if f covariant
  $0 \to f(A) \to f(B) \to f(C) \to 0$ is exact.

— case of $\mathrm{Hom}_{\mathbb{Z}}$:

Note: If $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ is exact, then exercise:

ex: $0 \to \underset{(\mathbb{Z})}{\mathrm{Hom}}(A'', B) \xrightarrow{j^*} \mathrm{Hom}(A,B) \xrightarrow{i^*} \mathrm{Hom}(A', B)$ is exact, but

i^* need not be surjective, i.e., not every map $A' \to B$ is the restriction of a map $A \to B$, i.e., not every map $A' \to B$ extends to $A \to B$. (n.b. Ext stands for "extension").

counter-ex: $0 \to \underset{A'}{\mathbb{Z}} \xrightarrow{\times 2} \underset{A}{\mathbb{Z}} \to \mathbb{Z}/2 \to 0$ induces

$$\underset{\mathbb{Z}}{\underset{\parallel}{\mathrm{Hom}(A,B)}} \xrightarrow{\times 2} \underset{\mathbb{Z}}{\underset{\parallel}{\mathrm{Hom}(A',B)}}$$

, which is not surjective.

Rmk: If $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ is split SES, then in fact get a SES

$$0 \to \mathrm{Hom}(A'', B) \xrightarrow{j^*} \mathrm{Hom}(A,B) \xrightarrow{i^*} \mathrm{Hom}(A', B) \to 0.$$

$\wr\wr$ by splitting

$\mathrm{Hom}(A' \oplus A'', B)$

easy to see this map surjects ("extend by 0").

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Similarly, if $0 \to A' \xrightarrow{i} A \to A'' \to 0$ exact, then we are guaranteed only that

$$A' \otimes_{\mathbb{Z}} B \xrightarrow{i \otimes \mathrm{id}_B} A \otimes_{\mathbb{Z}} B \to A'' \otimes_{\mathbb{Z}} B \to 0$$

is exact; $i \otimes \mathrm{id}_B$ need not be injective.

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How to measure these "failures of exactness"?

Use projective (or injective) resolutions as a 'replacement' of our given group/module.

(particularly nice modules for which the above problems don't arise)

Def:

An R-module Q is an injective R-module if, for any injective map (of R-modules) $f: M \to N$,

and any map $g: M \to Q$ $\exists$

$$\begin{tikzcd} (\text{SES}) \quad 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"] & N \arrow[dl, dashed] \\ & Q & \end{tikzcd}$$

$\exists\, h$ with $h \circ f = g$.

"any g to Q extends along injections".

(injective) Q

$\overset{\text{exercise}}{\iff}$ if any SES $0 \to Q \to M \to K \to 0$ splits.

$\overset{\text{exercise}}{\iff}$ if $\mathrm{Hom}_R(-, Q)$ is exact.

<u>Defs</u> An R-module P is <u>projective</u> if for any surjection $f: N \to M$ and any map $g: P \to M$, $\exists$:

$$\begin{tikzcd} N \arrow[r, "f"] & M \arrow[r] & 0 \\ & P \arrow[u, "g"] \arrow[ul, dashed, "\exists h"] & \end{tikzcd} \quad \text{(SES)}$$

with $f \circ h = g$. (projective)

"any g from P lifts along surjections."

$\iff$ any $\overset{\text{SES}}{0 \to A \to B \to P \to 0}$ is split.

$\iff$ $\mathrm{Hom}(P, -)$ is exact.

<u>Thm</u>: (exercise or look up in a book): For a $\mathbb{Z}$-module M (i.e., an abelian group) (or more generally M over a <u>PID</u> R).

- M is injective iff it is <u>divisible</u>.
  (an abelian group G is divisible if for any $g \in G$ and any $n \in \mathbb{N}$, $g = n \cdot g'$ for some $g' \in G$)
  <u>ex</u>: $\mathbb{Q}$, <u>non-ex</u>: $\mathbb{Z}$, or $\mathbb{Z}/2$
  but $\mathbb{Z}/2$ is injective as a $\mathbb{Z}/2$-module!

- M is projective iff it is free.

<u>Cor</u>: For a projective $\mathbb{Z}$-module P, given any injection $0 \to P' \to P$ (i.e., $\hookrightarrow$ a submodule), P' is projective too. (subgroups of free abelian groups are free abelian)

Similarly, if Q injective $\mathbb{Z}$-module, $Q \to Q' \to 0$, Q' injective too.

A <u>projective resolution</u> of an R-module M is an <u>exact</u> sequence

$$\to \cdots \to P_2 \to P_1 \to M \to 0$$

(resp. <u>injective</u> resolution:

$$0 \to Q \to I_1 \to I_2 \to \cdots).$$

It turns out these resolutions always exist, and

as a consequence of Thm / Cor above we deduce:

Any $\mathbb{Z}$-module (i.e., abelian group) M has a two-step projective (resp. injective) resolution

$$0 \to P_1 \to P_0 \xrightarrow{f} M \to 0.$$ (why? Pick surjection $f: P_0 \to M$ & have $0 \to \ker f \to P_0 \to M \to 0$; $\ker f$ proj. by cor)

(resp. $0 \to M \to I_0 \to I_1 \to 0$).

Def: Given R-modules M, N, pick projective resolution of M

$$(\cdots \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0) \to M \to 0,$$

take $\mathrm{Hom}_R(-, N)$ of all P_i's & f_i's between them (but not of M):

$$\cdots \xleftarrow{f_1^*} \mathrm{Hom}(P_1, N) \xleftarrow{f_0^*} \mathrm{Hom}(P_0, N) \leftarrow 0$$

This is no longer exact; however it's a chain complex: $f_{i+1}^* \circ f_i^* = 0$. (so $\ker f_{i+1}^* \supset \mathrm{im} f_i^*$).

We define $\mathrm{Ext}^i_R(M,N) := \dfrac{\ker f_i^*}{\mathrm{im} f_{i-1}^*}$. Note $\mathrm{Ext}^0_R(M,N) = \ker f_0^* \cong \mathrm{Hom}_R(M,N)$

(b/c $P_1 \xrightarrow{f_0} P_0 \to M \to 0$ induces an exact sequence $0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(P_0,N) \xrightarrow{f_0^*} \mathrm{Hom}(P_1,N)$)

This a priori depends on a choice (of proj. resolution);

however

Thm: $\mathrm{Ext}^i_R(M,N)$ are independent of choices made and functorial in M/N.

Rmk: Can also define $\mathrm{Ext}^i_R(M,N)$ by injectively resolving N, exercise that the result is unchanged.

Cor: For $\mathbb{Z}$-modules M, N, $\mathrm{Ext}^i_{\mathbb{Z}}(M,N) = 0$ for $i \geq 2$.

(b/c M admits a two-term proj. resolution $0 \to P_1 \to P_0 \to M \to 0$).

<u>Example computations</u> (using theorem): let's compute, for abelian groups H, G ($\mathbb{Z}$-modules), $\mathrm{Ext}^{(1)}(H, G)$, where

- H free (i.e., projective): can use resolution: $0 \to \underset{0}{P_1} \to \underset{H}{P_0} \to H \to 0$

$\Rightarrow$ Ext complex is:

$$\underset{\deg 1}{0} \leftarrow 0 \leftarrow \underset{\deg 0}{\mathrm{Hom}(P_0, G)} \leftarrow 0 \qquad \mathrm{Hom}(P_0,G) = \mathrm{Hom}(H,G)$$

$\Rightarrow \mathrm{Ext}(H,G) = 0.$

- $H = \mathbb{Z}/n\mathbb{Z}$, use

$$0 \to \big(\underset{\underset{P_1}{\shortparallel}}{\mathbb{Z}} \underset{f}{\xrightarrow{\times n}} \underset{\underset{P_0}{\shortparallel}}{\mathbb{Z}}\big) \to \mathbb{Z}/n\mathbb{Z} \to 0$$

$\rightsquigarrow$ Ext $\overset{\text{chain}}{\text{complex}}$: $0 \leftarrow \overset{G}{\overset{\shortparallel}{\mathrm{Hom}(\mathbb{Z},G)}} \underset{\delta^0 = f^*}{\overset{\times n}{\longleftarrow}} \overset{G}{\overset{\shortparallel}{\mathrm{Hom}(\mathbb{Z},G)}} \leftarrow 0$

$\Rightarrow \mathrm{Ext}(H,G) = \mathrm{coker}(\delta^0) = G/nG$.

$\begin{cases} \bullet\ \mathrm{Ext}(\mathbb{Z}/n, \overset{G}{\overset{\shortparallel}{\mathbb{Z}}}) = \mathbb{Z}/n \\ \bullet\ \mathrm{Ext}(\mathbb{Z}/n, \mathbb{Q}) = 0 \end{cases}$

- $H = H_1 \oplus H_2$, then we can add projective resolutions of H_1 & H_2 to get one for H

  $\Rightarrow \mathrm{Ext}(H_1 \oplus H_2, G) = \mathrm{Ext}(H_1, G) \oplus \mathrm{Ext}(H_2, G)$.

Similarly, to define $\mathrm{Tor}_j^R(M,N)$, projectively resolve either M or N:

$\underbrace{(\cdots \to P_1 \to P_0)}_{P_\bullet} \to M \to 0$, then take $P_\bullet \otimes_R N$; again the result is a chain complex (no longer exact); jth homology is $\mathrm{Tor}_j^R(M,N)$.

& same Thm: functorial + independent of choice.

Sketch of proof of theorem on independence of Ext (resp. Tor) from choices:

All follows from "$\overset{\text{homotopical}}{\text{functoriality}}$ of projective resolutions":

Thm: Say $P_\bullet \to M$ is a proj. resolution and $\bar{f}: M \to N$ is a map (of R-modules), and $Q_\bullet \to N$ a proj. resolution. Then there is a $\overset{\text{chain}}{\text{map}}$ $f_\bullet: P_\bullet \to Q_\bullet$ 'lifting' $\bar{f}$, meaning:

$$\begin{tikzcd} \cdots \arrow[r, "\partial^P"] \arrow[d, dashed] & P_1 \arrow[r, "\partial^P"] \arrow[d, dashed, "\exists f_2"] & P_0 \arrow[r] \arrow[d, dashed, "\exists f_1"] & M \arrow[r] \arrow[d, "\bar{f}"] & 0 \\ \cdots \arrow[r, "\partial^Q"'] & Q_1 \arrow[r, "\partial^Q"'] & Q_0 & N \arrow[r] & 0 \end{tikzcd}$$

making diagram commute.

and f is unique up to homotopy equivalence (meaning if f, f' two lifts then $\exists\ h: P_\bullet \to Q_{\bullet+1}$ with $f - f' = \partial^Q \circ h + h \circ \partial^P$).

— This induces, for $f: M \to N$, a map f^*: between $\overset{\text{co-chain}}{\text{complexes}}$ computing $\mathrm{Ext}_R^\bullet(M,S)$ & $\mathrm{Ext}_R^\bullet(M,S)$.

- To see ind. of choices $\overset{\text{for this,}}{}$ let $N = M$, $\bar{f} = \mathrm{id}$, $P_\bullet$, $Q_\bullet$ two different resolutions

  $\Rightarrow$ get a $\overset{\text{chain}}{\text{map}}$ $f: P_\bullet \to Q_\bullet$ lifting id, & hence (by taking $\mathrm{Hom}(-,S)$)

a chain map between Ext complexes, f^*

we also get a map $g: Q_\bullet \to P_\bullet$ & get g^* other way between Ext complexes:

By homotopical uniqueness, $f\circ g$ & $g \circ f$ are each homotopic to $id_{Q_\bullet}$, $id_{P_\bullet}$ respectively, hence f^*, g^* induce isomorphisms on Ext groups

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Sketch of proof of theorem: idea is to inductively use the projectivity condition:

inductively say we've constructed $f_i, \dots, f_0$. (base case: $\bar{f}$ itself $i=-1$)

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & P_{i+1} \arrow[r, "\partial^P"] \arrow[d, dashed, "\exists ?\ f_{i+1}?"] & P_i \arrow[r, "\partial_i^P"] \arrow[d, "f_i"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[d, "\bar{f}"] \\
\cdots \arrow[r, dashed] & Q_{i+1} \arrow[r, "\partial^Q"'] & Q_i \arrow[r, "\partial^Q"] & \cdots \arrow[r] & Q_0 \arrow[r] & N
\end{tikzcd}$$

want: $\partial^Q \circ f_{i+1} = f_i \circ \partial^P$. Since sequences at top and bottom are exact, we can restrict to:

$$\begin{tikzcd}
P_{i+1} \arrow[r] \arrow[d, dashed, "\exists \text{ by projectivity.}"'] \arrow[dr, bend left, "\text{map from proj.}"] & \underbrace{\ker(\partial_i^P)}_{\operatorname{im}(\partial_{i+1}^P)} \arrow[r] \arrow[d, "f_i"] & 0 \\
Q_{i+1} \arrow[r, "\text{surjective}"'] & \ker(\partial_i^Q) \arrow[r] & 0
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
0 \arrow[r] & \ker(\partial_i^P) \arrow[r] \arrow[d, "f_i"] & P_i \arrow[r] \arrow[d, "f_i"] & \cdots \arrow[d, "f_{i-1}"] \\
0 \arrow[r] & \ker(\partial_i^Q) \arrow[r] & Q_i \arrow[r] & \cdots
\end{tikzcd}$$

Uniqueness up to homotopy? exercise. Similar inductive level-by-level construction of

$$\begin{tikzcd}
\cdots \arrow[r] & P_{i+1} \arrow[r] \arrow[d, "f_{i+1}-f_{i+1}'"] \arrow[dl, "h_{i+1}"'] & P_i \arrow[r] \arrow[d, "f_i-f_i'"] \arrow[dl, "h_i"'] & \cdots \\
\cdots \arrow[r] & Q_{i+1} \arrow[r] & Q_i \arrow[r] & \cdots
\end{tikzcd}$$

satisfying $\partial h + h \partial = f - f'$.

☒

Back to the universal coefficient theorem (for cohomology, for simplicity): $C_\bullet$ free chain complex

wanted:
$$0 \to \operatorname{Ext}(H_{n-1}(C_\bullet), G) \to H^n(\operatorname{Hom}_{\mathbb{Z}}(C_\bullet, G)) \xrightarrow{\beta} \operatorname{Hom}_{\mathbb{Z}}(H_n(C_\bullet), G) \to 0$$

where $\beta([f])([c]) = f(c)$. ($\exists$ lift $\dashrightarrow$ s)

Introduce some notation: $C_\bullet = \{ \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \}$

$Z_n \subset C_n$ cycles $Z_n = \ker \partial_n$

$B_n \subset C_n$ boundaries $B_n = \mathrm{im}(\partial_{n+1})$, so have two relevant SES:

(1) $0 \to B_n \to Z_n \to H_n \to 0$ (by def'n of H_n)

(proj. resolution) $\uparrow H_n(C_\bullet)$

(2) $0 \to Z_n \xrightarrow{\text{incl.}} C_n \xrightarrow{\partial_n} B_{n-1} \to 0$ (by for any $\partial_n : A \to B$: SES $0 \to \ker\partial_n \to A \to \mathrm{im}\,\partial_n \to 0$)

Lem: $Z \subset C$ Abelian groups, and say C/Z is free. Then every homomorphism $t: Z \to G$ extends to $\hat{t}: C \to G$. ∎.

(restatement of the fact that if $0 \to Z \to C \xrightarrow{\partial} C/Z \to 0$ w/ C/Z projective, then $\mathrm{Hom}(-, G)$ gives a SES $\Longleftrightarrow$ the SES is split).

β is surjective:

Take $s \in \mathrm{Hom}_{\mathbb{Z}}(H_n, G)$, so $s: \dfrac{Z_n}{B_n} \to G$, this induces

$$\begin{tikzcd} Z_n \arrow[rr, "t"] \arrow[dr] & & G \\ & Z_n/B_n \arrow[ur, "s"'] & \end{tikzcd}$$

Note: since $B_n \subset Z_n \subset C_n$ (free), B_{n-1} & Z_n are free.

So Lem and (2) $\Rightarrow t$ extends to $a: C_n \to G$, and $a|_{B_n} = t|_{B_n} = 0$.

Thus if $a(\partial(-)) = 0$, $\Rightarrow \delta a = 0$, so $[a] \in H^n(\mathrm{Hom}(C_\bullet, G))$ lifts s. Hence β is surjective.

Failure of β to be injective:

Say $[a] \in \ker(\beta) \subset H^n(\mathrm{Hom}(C_\bullet, G))$, a is a cocycle, meaning $a|_{B_n} \equiv 0$.

Being in $\ker(\beta)$ means $a(c) = 0$ for all cycles c, so $a|_{Z_n} \equiv 0$.

So a in that case induces

$$\frac{C_n}{Z_n} \to G.$$

11/2 (2)

$\bar{a} : B_{n-1}$.

Given by $\bar{a}(\partial\alpha) := a(\alpha)$, $\bar{a} \in \mathrm{Hom}(B_{n-1}, G)$.

Change a by a coboundary?

Suppose $a = \delta b$: (difference between two representatives), $b \in \mathrm{Hom}(C_{n-1}, G)$

Then, $\bar{a}(\partial\alpha) = a(\alpha) = \delta b(\alpha) = b(\partial\alpha)$, i.e.,

$\bar{a} = b|_{B_{n-1}}$, where $b : C_{n-1} \to G$.

Claim: a is a coboundary if and only if $\bar{a}$ extends to C_{n-1}.

$\Rightarrow$ above.

$\Leftarrow$: Say $\bar{a}(\partial\alpha) = b(\partial\alpha)$, $b : C_{n-1} \to G$.

$$\begin{array}{ccc} \bar{a}(\partial\alpha) & = & b(\partial\alpha) \\ \| & & \| \\ a(\alpha) & & \delta b(\alpha) \end{array}$$

$\Rightarrow a = \delta b$.

In fact, a is a coboundary $\Leftrightarrow$ $\bar{a}$ extends to Z_{n-1}

(b/c by Lemma, since $0 \to Z_{n-1} \to C_{n-1} \to \underset{\text{free}}{B_{n-2}} \to 0$, any map out of Z_{n-1} automatically extends to C_{n-1}).

Get: a well-defined map

$$[a] \longmapsto \bar{a}$$

$$\ker(\beta) \xrightarrow{(*)} \underline{\mathrm{Hom}(B_{n-1}, G)}$$

"those Homs $B_{n-1} \to G$ that extend to $Z_{n-1} \to G$."

this is $\mathrm{Ext}(H_{n-1}, G)$, by below!

well-defined by $\Rightarrow$, and injective by $\Leftarrow$.

(Observe: $0 \to (B_{n-1} \xrightarrow{i} Z_{n-1}) \to H_{n-1} \to 0$ gives a projective resolution of H_{n-1}!

$P_\bullet$ = $(B_{n-1} \xrightarrow{i} Z_{n-1})$

Taking $\mathrm{Hom}(P_\bullet, G)$: we get

$0 \leftarrow \mathrm{Hom}(B_{n-1}, G) \xleftarrow{i^*} \mathrm{Hom}(Z_{n-1}, G) \leftarrow 0$, and $\mathrm{Ext}(H_{n-1}, G) := \mathrm{coker}(i^*) = \mathrm{Hom}(B_{n-1}, G) / i^* \mathrm{Hom}(Z_{n-1}, G)$

Surjectivity of (☆): Given $\Theta : B_{n-1} \to G$, define

$$a : C_n \xrightarrow{\partial} B_{n-1} \xrightarrow{\theta} G \quad , \text{ i.e., } a(\alpha) = \Theta(\beta\alpha)$$

a by definition annihilates $Z_n \Rightarrow$ annihilates $B_n \Rightarrow a$ cocycle.

And moreover $[a] \in \ker \beta$ with $[a] \mapsto \Theta$.

So in conclusion, get a SES:

$$0 \to E = \ker(\beta) \to H^n(\mathrm{Hom}(C_\bullet, G)) \to \mathrm{Hom}(H_n(C_\bullet), G) \to 0.$$

$$E \cong \mathrm{Hom}(B_{n-1}, G) / i^*\mathrm{Hom}(Z_{n-1}, G) = \mathrm{Ext}(H_{n-1}, G)$$

$\boxtimes$.

This sequence further splits (continued).