Last time:

We gave a construction of Chern classes of a complex vector bundle (resp. Stiefel-Whitney classes of a real vec. bundle), using Leray-Hirsch theorem.

(To recap: $E \to B$ cplx vec. bundle of rank k, $\rightsquigarrow$ $\mathbb{P}(E) \to B$ fiberwise cplx projectivize, & $\exists$ canonical coh. class $h_p \in H^2(\mathbb{P}(E);\mathbb{Z})$. $\left(= -c_1^{old}\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{P}(E)\end{smallmatrix}\right)\right)$. The Chern classes $c_i(E) \in H^{2i}(B;\mathbb{Z})$ are the unique classes a_i s.t.

$$h_p^k + \pi^*(a_1) \cup h_p^{k-1} + \dots + \pi^*(a_k) \cup h_p^0 = 0.)$$

. (analogously for w_i).

We checked: Whitney sum formula, naturality, also $c_1 = c_1^{old}$; in particular $c_1\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{C}\mathbb{P}^\infty\end{smallmatrix}\right) = -h \in H^2(\mathbb{C}\mathbb{P}^\infty;\mathbb{Z})$.

We want to show the classes c_i are axiomatically determined by

☆
- naturality
- Whitney sum formula.
- dimension $c_i(E) = 0$ for $i > \mathrm{rank}_{\mathbb{C}}(E)$
- (normalization) $c_1\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{C}\mathbb{P}^\infty\end{smallmatrix}\right) = -h$.

This would complete the axiomatic characterization of Chern classes. (resp. same for Stiefel-Whitney).

Uniqueness? We have classes c_i as constructed above. Say we are given $\tilde{c}_1, \dots, \tilde{c}_i$ other char. classes which satisfy the axioms ☆.

Since $\tilde{c}_1\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{C}\mathbb{P}^\infty\end{smallmatrix}\right) = -h = c_1\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{C}\mathbb{P}^\infty\end{smallmatrix}\right)$, and $\tilde{c}_i\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{C}\mathbb{P}^\infty\end{smallmatrix}\right) = 0 = c_i\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{C}\mathbb{P}^\infty\end{smallmatrix}\right)$ for $i > 1$,

$\Rightarrow$ $\tilde{c}_i = c_i$ for all i for $\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{C}\mathbb{P}^\infty\end{smallmatrix}$ $\underset{\text{(naturality)}}{\Longrightarrow}$ $c_i = \tilde{c}_i$ for all i for all complex line bundles $\begin{smallmatrix} L \\ \downarrow \\ B\end{smallmatrix}$.

(of course $c_i = \tilde{c}_i = 0$ if $i > 1$).
Content: $c_1(L) = \tilde{c}_1(L)$.

$$\Rightarrow \quad c(L) = \tilde{c}(L) \quad \text{where} \quad c = \overset{c_0 \text{ by convention}}{\overbrace{1}} + c_1 + c_2 + \dots \qquad \tilde{c} = \overset{\text{by convention } \tilde{c}_0.}{\overbrace{1}} + \tilde{c}_1 + \dots . \quad \text{'total Chern class'}$$

$\Longrightarrow$ If a complex line bundle E can be written as a direct sum $E = L_1 \oplus \dots \oplus L_k$ of line bundles, then Whitney sum formula implies:

$$c(E) = 1 + c_1(E) + \dots \underset{\substack{\text{Whitney} \\ \text{sum for} \\ c}}{=} \prod_{i=1}^{k} c(L_i) \underset{\substack{\text{by} \\ \text{above} \\ c(L_i) = \tilde{c}(L_i) \text{ for line bundles}}}{=} \prod \tilde{c}(L_i) \underset{\substack{\text{Whitney} \\ \text{sum for} \\ \tilde{c}}}{=} \tilde{c}(E).$$

Problem: A given vector bundle E need not admit such a decomposition.

(e.g., over S^4, the clutching construction tells us that $\text{Vect}_2^{\mathbb{C}}(S^4) \underset{\text{existence of metrics}}{\cong} \text{Vect}_2^{\text{Hermitian},\mathbb{C}}(S^4)$

$\cong [S^3, U(2)] \underset{\text{direct computation, omit.}}{\cong} \mathbb{Z}$, i.e., ∃ non-trivial rank-2 cplx vec. bundles

structure group for a Hermitian rank 2 bundle.

On the other hand, we've previously seen that $\text{Vect}_1^{\mathbb{C}}(S^4) = [S^3, S^1 = U(1)] = \{*\}$.
(by same argument)

So a non-trivial $\overset{\text{rank } 2}{E} \longrightarrow S^4$ doesn't decompose).

However, we can appeal to the following powerful principle:

Prop: (Splitting principle) (we'll state for cplx vec bundles, but real case analogous w/ 'same' proof).

Given any X (paracompact), any complex v.b. $E \to X$, ∃ a space Z and a map $s: Z \to X$ such that

(a) $s^*E \longrightarrow Z$ is isomorphic to a direct sum of line bundles.

(b) $s^*: H^*(X;\mathbb{Z}) \longrightarrow H^*(Z;\mathbb{Z})$ is injective.

(statement for real vector bundles involves injectivity of s^* on $H^*(-;\mathbb{Z}/2)$).

Using the splitting principle: Say E any rank k vector bundle $\to B$. Fix an $s: Z \to B$ as in splitting principle, so $s^*E \cong L_1 \oplus \cdots \oplus L_k$. Then, we see that if $\{c_i\}$, $\{\tilde{c}_i\}$ any two systems of "chern classes" (satisfying axioms), then:

$$\begin{tikzcd} \tilde{c}_i(s^*E) \arrow[r, equal] \arrow[d, equal, "\text{naturality}"] & c_i(s^*E) \arrow[d, equal, "\text{naturality}"] \\ s^*\tilde{c}_i(E) & s^*c_i(E) \end{tikzcd}$$

by arguments above, b/c $c_i = \tilde{c}_i$ on any vector bundle which splits into line bundles, & s^*E splits.

We learn $s^*\tilde{c}_i(E) = s^*c_i(E)$. Since s^* is injective, $\Rightarrow \tilde{c}_i(E) = c_i(E)$. Uniqueness ✓.

First, a quick observation: If
$$\begin{tikzcd} F \arrow[r, phantom, "\subset"] \arrow[dr] & E \arrow[d] \\ & X \end{tikzcd}$$
vector subbundle of E, then using a fiberwise (Hermitian) metric structure

always exists if X paracompact.

(i.e., a 'continuous' family of $\langle -,- \rangle_p$ on E_p's) can define the orthogonal complement of a subbundle.

$F^\perp \subset E$ by $(F^\perp)_p := (F_p)^\perp$ using $\langle -,- \rangle_p$ on E_p.

depends on metric

This is a vector sub-bundle of E, complementary to F in each fiber $\Rightarrow$ get an iso. of vector bundles

$E \cong F \oplus F^{\perp} \overset{\cdot}{\cong} F \oplus E/F$. , i.e., $E \cong F \oplus \underset{\nwarrow}{E/F}$.

this bundle is defined w/o $\langle -,- \rangle_p$ but $\overset{\cdot}{\cong}$ uses $\langle -,- \rangle_p$.

__Pf of splitting principle:__

By induction on $\text{rank}_{\mathbb{C}}(E)$:

- true when $\text{rank}_{\mathbb{C}}(E) = 1$. ✓.
- general case of rank k (assuming true for all rank $(k-1)$ vec. bundles on all paracompact spaces):

$$E \to X \quad \text{rank } k. \quad \text{Let } Z_1 = \underset{\uparrow}{\mathbb{P}(E)} \xrightarrow{s_1 = \pi} X \qquad (\text{fibers are } \mathbb{C}P^{k-1}\text{'s})$$

fiberwise complex projectivization.

Recall that Leray-Hirsch applies to $\mathbb{P}(E)$ using coh. extension of fiber given by $(\underset{\uparrow}{\textcircled{1}}, h_p, \_\_\_)$

$\Rightarrow \pi^* = s_1^* : H^\bullet(X) \longrightarrow H^\bullet(\mathbb{P}(E))$ is injective.

(b/c $H^\bullet(\mathbb{P}(E))$ is freely gen. as a $H^\bullet(X)$-module (module str. comes from s_1^* & $\cup$)
by 1, other classes).

Looking at $\tilde{E} = s_1^* E \longrightarrow \mathbb{P}(E)$; the fiber at a point $(\overset{X}{\overset{\cup}{x}}, \overset{E_x}{\overset{\cup}{\ell}}) \in \mathbb{P}(E)$ is E_x.

In particular, the tautological line bundle $L_{taut} \to \mathbb{P}(E)$ is actually a vector sub-bundle of $\tilde{E}$:

$$\underset{\substack{\uparrow \\ \text{fiber over } (x, \ell \subseteq E_x) \text{ is} \\ \ell}}{L_{taut}} \quad \underset{\subseteq}{\subseteq} \quad \underset{\substack{\uparrow \\ \text{fiber over } (x, \ell \subseteq E_x) \text{ is} \\ E_x}}{\tilde{E}}$$

By observation right above the proof, paracompactness $\Rightarrow$ (using metric structure, e.g.,)

can split $\quad \tilde{E} = L_{taut} \oplus \underbrace{E_1}$.

cplx. vector bundle over $Z_1 := \mathbb{P}(E)$ of rank $(k-1)$.

By inductive hypothesis, $\exists \quad s_2 : Z \to Z_1$ w/ s_2^* injective on cohomology

and $\quad s_2^* E_1 \cong L_2 \oplus \cdots \oplus L_k$

$\Rightarrow s := s_1 \circ s_2 : Z \longrightarrow Z_1 \longrightarrow X$ satisfies:

- $s^* = s_2^* s_1^*$ injective on $H^\bullet(-;\mathbb{Z})$

- $s^*E = s_2^*(s_1^*E) = s_2^*\tilde{E} = s_2^*(L_{taut} \oplus E_1)$

$$\cong \underbrace{L_1}_{\overset{\cdot\cdot}{=}\, s_2^* L_{taut}.} \oplus L_2 \oplus \cdots \oplus L_k \qquad \blacksquare.$$

Unwinding the induction, we can spell out what the final Z is:

$$\begin{tikzcd} Z \arrow[rrr, bend left, "s"] & \cdots \arrow[r] & Z_2 \arrow[r] \arrow[d, equal] & Z_1 \arrow[r] \arrow[d, equal] & X \\ & & \mathbb{P}(L_{taut}^{\perp}) & \mathbb{P}(E) & \end{tikzcd}$$

in $s_1^*(E) = \tilde{E}$ over $\mathbb{P}(E)$. $(E_1 = L_{taut}^{\perp}$ for some metric on $E)$

Point in the fiber of Z_2 over $(x,\ell) \in \mathbb{P}(E) = Z_1$ is a line $L_2 \subseteq L_1^{\perp} = E_1 \subseteq E_x$.

<u>Thus</u>: If we use a fixed Hermitian metric on E (inducing one on all its pullbacks & sub-bundles)

$$s: Z \to X \text{ has fiber over } x \in X \text{ equal to } \left\{ (L_1, \dots, L_k) \;\middle|\; \begin{array}{l} L_i \subseteq E_x \text{ line} \\ L_i \perp L_j \text{ for } i \neq j \text{ using } \langle -,- \rangle_x \\ \Rightarrow L_1 \oplus \cdots \oplus L_k = E_x \end{array} \right\}$$

If V cplx. vec. space dimension n w/ inner product, the <u>complex flag manifold</u>

$$\mathbb{F}(V) = \{(\ell_1, \dots, \ell_n) \mid \ell_i \perp \ell_j\} \qquad V_i := \ell_1 \oplus \cdots \oplus \ell_i.$$

w/o an inner product, can still describe as $\mathbb{F}(V) = \{(V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n)$ w/ $\dim V_i = i\}$.

($\mathbb{F}(V)$ & $\mathbb{P}(V)$ sit within a wide collection of (generalized) flag manifold)

Above, we see that $s: Z \to X$ is a fiber bundle w/ fiber $\mathbb{F}(E_x)$.

As mentioned, everything above works for real vec. bundles as well, using $Z :=$ real version of $\mathbb{F}(E) \longrightarrow X$.

$\Rightarrow$ uniqueness of Stiefel-Whitney classes given the axioms.

<u>Some computations</u> (starting with Stiefel-Whitney classes):

Smooth manifolds come equipped w/ a natural vector bundle, their tangent bundle $\begin{array}{c} TM \\ \downarrow \\ M^m \end{array}$. We can use

real rank m vec. bundle

dimension

the axioms to compute $w_i(TM) \underset{\text{shorthand}}{=:} \boxed{w_i(M).}$

<u>Ex</u>: $S^n \subseteq \mathbb{R}^{n+1}$ unit sphere.

Recall that we can explicitly define $T_x S^n = \{v \in \mathbb{R}^{n+1} \mid v \overset{\text{using } \langle -,- \rangle_{\text{Euclidean}}}{\perp} x\}$

In particular, $T_xS^n \subseteq \mathbb{R}^{n+1}$ inducing $TS^n \subseteq \underline{\mathbb{R}}^{n+1}$ ← trivial bundle over S^n, & moreover there's a direct sum decomposition

$$T_xS^n \oplus \mathbb{R} \xrightarrow{\cong} \mathbb{R}^{n+1} \qquad \text{inducing an iso. } TS^n \oplus \underline{\mathbb{R}} \xrightarrow{\cong} \underline{\mathbb{R}}^{n+1}.$$
$$(v, t) \longmapsto \underset{\substack{\uparrow \\ \text{orthogonal to } x.}}{v} + tx.$$

By Whitney sum formula,

$$\Rightarrow w(TS^n) \cup \underbrace{w(\underline{\mathbb{R}})}_{\substack{\shortparallel \\ 1 = w_0}} = \underbrace{w(\underline{\mathbb{R}}^{n+1})}_{\substack{\shortparallel \\ 1 = w_0.}}$$

$$\Rightarrow \overset{\substack{H^0 \\ \downarrow}}{1} + \overset{\substack{H^1(S^n;\mathbb{Z}/2) \\ \downarrow}}{w_1(S^n)} + \overset{\substack{H^2 \\ \downarrow}}{w_2(S^n)} + \cdots = 1.$$

$$\Rightarrow w_i(S^n)\left(\overset{\text{def}}{=} w_i(TS^n)\right) = 0 \text{ for all } i > 0.$$

Note TS^n is not always trivial! (HW exercise, e.g., TS^{2k} has no non-vanishing sections).

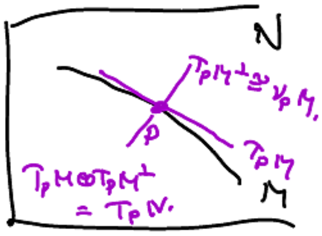
(In gen'l, say a vector bundle $\overset{E \text{ (rank } k)}{}$ is stably trivial if

$$E \oplus \underline{\mathbb{R}}^{\ell} \cong \underline{\mathbb{R}}^{k+\ell} \text{ for some } \ell.$$

Whitney sum formula as above ⇒ stably trivial E have $w_i(E) = 0$ for all i so w_i not a complete invariant).

In gen'l, we can study submanifolds $M^m \overset{i}{\subset} N^n$ via characteristic classes using the fact that

$$\underbrace{TN|_M}_{\substack{\text{same as } i^*TN \\ \downarrow \\ M}} \underset{\substack{\uparrow \\ \text{using} \\ \text{a metric}}}{\cong} \underset{\substack{\uparrow \\ \text{sub-bundle} \\ \text{of } TN|_M}}{TM} \oplus (TM)^{\perp} \cong TM \oplus \underset{\substack{\uparrow \\ \text{normal bundle to } M \subseteq N \\ := TN|_M / TM.}}{\nu M}$$



More on this next time.

Let's understand $T\overbrace{Gr_k(\mathbb{R}^n)}^{k\text{-planes in } \mathbb{R}^n}$. To understand Tangent bundle, first understand manifold structure.

Let $E_0 \in Gr_k(\mathbb{R}^n)$ any point (i.e., $E_0 \subseteq \mathbb{R}^n$ k-dim'l). So $\mathbb{R}^n = E_0 \oplus E_0^{\perp}$ ← using $\langle -,- \rangle_{\text{Eucl.}}$

consider the map

$$\psi_{E_0}: \operatorname{Hom}_{\mathbb{R}}(E_0, E_0^{\perp}) \longrightarrow Gr_k(\mathbb{R}^n)$$
$$\overset{\Downarrow}{a} \longmapsto \text{graph of } a = (\overbrace{id \oplus a}^{E_0 \nearrow E_0 \oplus E_0^{\perp}})(E_0) \underset{\substack{k\text{-dim'l} \\ (\text{b/c } id\oplus a \text{ injective})}}{\subseteq} E_0 \oplus E_0^{\perp} \cong \mathbb{R}^n.$$

Claim (exercise): Image of ψ_{E_0} is an open nbhd of E_0, U_{E_0}

the maps $\psi_{E_0}^{-1}: U_{E_0} \longrightarrow \operatorname{Hom}_{\mathbb{R}}(E_0, E_0^{\perp}) \underset{\text{linearly}}{\cong} \mathbb{R}^{k(n-k)}$ makes $Gr_k(\mathbb{R}^n)$ into a smooth $k(n-k)$-dim'l manifold.

The tangent space at $E_0 \in Gr_k(\mathbb{R}^n)$ is isomorphic to $\operatorname{Hom}_{\mathbb{R}}(E_0, E_0^{+})$:

$$\begin{tikzcd} \bigstar \quad d(\psi_{E_0})_0 : T_0 \underline{\mathrm{Hom}}_{\mathbb{R}}(E_0, E_0^{\perp}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & T_{E_0} Gr_k(\mathbb{R}^n). \\ \mathrm{Hom}_{\mathbb{R}}(E_0, E_0^{\perp}) & \end{tikzcd}$$

Globalizing, let $\begin{matrix} E_{taut} \\ \downarrow \\ Gr_k(\mathbb{R}^n) \end{matrix}$ the tautological rank k vec. bundle $(E_{taut})_{E_0} = E_0$; we have

(fiber at E_0 is E_0) $\subseteq$ (fiber at E_0 is $\mathbb{R}^n$).

$$\begin{tikzcd} E_{taut} \arrow[r, phantom, "\subseteq"] \arrow[dr] & \underline{\mathbb{R}}^n \arrow[d] \\ & Gr_k(\mathbb{R}^n) \end{tikzcd}$$

. Now using $\langle -,- \rangle_{Eucl}$ on $\underline{\mathbb{R}}^n$ we can split $\underline{\mathbb{R}}^n \cong E_{taut} \oplus E_{taut}^{\perp}$.

and there is an isomorphism of vector bundles $\underline{\mathrm{Hom}}(E_{taut}, E_{taut}^{\perp}) \xrightarrow{\cong} T\,Gr_k(\mathbb{R}^n)$ over $Gr_k(\mathbb{R}^n)$

$$(E_0, \underset{\substack{\underline{\mathrm{Hom}}(E_{taut}, E_{taut}^{\perp})_{E_0} \\ \| \\ \mathrm{Hom}(E_0, E_0^{\perp})}}{v \in}) \longmapsto (E_0, \underbrace{d(\psi_{E_0})_0(v)}_{\bigstar}).$$

(check: really a map of vector bundles, i.e., continuous).

Sub-example: $\mathbb{R}P^{n-1} = Gr_1(\mathbb{R}^n)$

$L = L_{taut}$ tautological line bundle. By above $T\mathbb{R}P^{n-1} = \underline{\mathrm{Hom}}_{\mathbb{R}}(L, L^{\perp})$,

$$\text{so} \quad T\mathbb{R}P^{n-1} \oplus \underline{\mathbb{R}} \cong \underline{\mathrm{Hom}}_{\mathbb{R}}(L, L^{\perp}) \oplus \underbrace{\underline{\mathbb{R}}}_{\substack{\cong \\ L^* \otimes L \,\cong\, \underline{\mathrm{Hom}}_{\mathbb{R}}(L,L)}}$$

(works only for line bundles)

$$\cong \underline{\mathrm{Hom}}_{\mathbb{R}}(L, L^{\perp} \oplus L)$$

$$\cong \underline{\mathrm{Hom}}_{\mathbb{R}}(L, \underset{\substack{\cong \\ \underbrace{\underline{\mathbb{R}} \oplus \dots \oplus \underline{\mathbb{R}}}_{n}}}{\underline{\mathbb{R}}^n}) \cong \bigoplus_{i=1}^{n} \underline{\mathrm{Hom}}_{\mathbb{R}}(L, \underline{\mathbb{R}}) \cong \underbrace{L^* \oplus \dots \oplus L^*}_{n \text{ copies.}}$$

$$\text{So,} \quad T\mathbb{R}P^{n-1} \oplus \underline{\mathbb{R}} \cong \underbrace{L^* \oplus \dots \oplus L^*}_{n \text{ copies.}}$$

This implies $w(T\mathbb{R}P^{n-1}) = w(\underbrace{L^* \oplus \dots \oplus L^*}_{n \text{ copies}})$ by Whitney sum formula. $\left(w(T\mathbb{R}P^{n-1}) \cup \overbrace{w(\underline{\mathbb{R}})}^{1} = w(L^{* \oplus n}) \right)$

We'll complete this next time.

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$L \to \mathbb{RP}^{n-1}$ tautological line bundle then $L^* \otimes L \cong \underline{\mathbb{R}}$ implies that

$w_1(L^*) + w_1(L) = 0$ (b/c $w_1(L \otimes L') = w_1(L) + w_1(L')$ — we showed this earlier in class — for line bundles)

$\Rightarrow w_1(L^*) = -w_1(L) = w_1(L) = h.$

(as w_1 is defined in $H^1(\mathbb{RP}^{n-1}; \underline{\mathbb{Z}/2})$).

So, $w(L^*) = 1+h$, so Whitney sum formula implies: in $H^*(\mathbb{RP}^{n-1};\mathbb{Z}/2) \cong \mathbb{Z}/2[h]/h^n$, $|h|=1$

$$w(\mathbb{RP}^{n-1}) := w(T\mathbb{RP}^{n-1}) = w((L^*)^{\oplus n}) = (1+h)^n$$

$$= 1 + \underset{\overset{\uparrow}{H^1}}{nh} + \binom{n}{2}\underset{\overset{\uparrow}{H^2}}{h^2} + \dots + nh^{n-1} + \cancel{h^n} \quad h^n = 0 \text{ in}$$

under the iso $H^i(\mathbb{RP}^{n-1};\mathbb{Z}/2)\cong \mathbb{Z}/2$ sending $h^i \mapsto 1$,

$$\Rightarrow \boxed{w_i(\mathbb{RP}^{n-1}) = \binom{n}{i} \bmod 2} \qquad \text{i.e., } \boxed{w_i(\mathbb{RP}^n) = \binom{n+1}{i} \bmod 2.}$$

Consequences:

Def: Say M^m is parallelizable if $TM \cong \underline{\mathbb{R}}^m \Longrightarrow w(M) := w(TM) = 1.$

The computation above reveals that

Cor: $\mathbb{RP}^n$ can only possibly be parallelizable if $n = 2^k - 1$.

(Pf: unless $n=2^k-1$ some k, $\exists$ i with $\binom{n+1}{i}$ odd, hence that $w_i(T\mathbb{RP}^n)\neq 0$).

Suppose $\mathbb{R}^{q+1}$ admits a bilinear product $\mathbb{R}^{q+1}\times\mathbb{R}^{q+1}\to\mathbb{R}^{q+1}$ w/o zero divisors;

when is this possible? (e.g., possible for $q=1$, using complex mult. $\mathbb{R}^2\times\mathbb{R}^2\cong\mathbb{C}\times\mathbb{C}\xrightarrow{*}\mathbb{C}\cong\mathbb{R}^2$).

Exercise: can prove that if $\mathbb{R}^{q+1}$ has such a mult, then $T\mathbb{RP}^q$ has q linearly independent sections & is therefore trivial; i.e., $\mathbb{RP}^q$ must be parallelizable.

Cor: $\mathbb{R}^{q+1}$ can only admit such a product if $q=2^k-1$.

(in fact more strongly only have such a product when $q = \underset{\text{reals}}{0}, \underset{\text{complex}}{1}, \underset{\text{quaternions}}{3}, \underset{\text{octonions}}{7}$, but these methods don't tell us that.)

Immersions + embeddings

If $f: M^m \longrightarrow N^n$ smooth map w/ $df_x : T_xM \to T_xN$ is injective $\forall\, x \in M$, say f is an immersion ($\Rightarrow \dim(N) \geq \dim(M)$). We say an immersion $f: M \to N$ is an embedding if further it is (proper) & injective.

not always required

immersion of $S^1 \to \mathbb{R}^2$ which is not an embedding

Special case of an embedding:

a submanifold $M \overset{i}{\subset} N$.

We can think of $\{df_x\}_x$ as inducing an (fiberwise) injective morphism of vector bundles:

$$\begin{tikzcd} TM \arrow[rr, hook, "df"] \arrow[dr] & & f^*TN \arrow[dl] \\ & M & \end{tikzcd}$$

(i.e., $df_x : T_xM \to (f^*TN)_x = T_{f(x)}N$.)

If $f = i : M \hookrightarrow N$ inclusion, $i^*TN = TN|_M$.

For any immersion $f: M \to N$ (including embeddings) there is an associated normal bundle

$$\begin{array}{c} \nu_M \\ \downarrow \\ M \end{array} \quad \text{defined by} \quad \nu_M := \frac{f^*TN}{\underbrace{df(TM)}_{\text{sub-bundle of } f^*TN}} \quad \begin{array}{c} \\ \downarrow \\ M \end{array}$$

← fiberwise quotient of vector bundles

(submanifold $M \subset N$: $\nu_M := TN|_M / TM$).

ν_M is a vector bundle of rank $n-m$, a choice of metric induces an isomorphism

$$f^*TN \cong df(TM) \oplus df(TM)^\perp$$
$$\cong TM \oplus \nu_M .$$

(sub-manifold: $TN|_M \cong TM \oplus \nu_M$).

This, plus Whitney sum formula, allows one to constrain/understand properties of embeddings & immersions provided one has control over TM, TN, (e.g., by telling us constraints on what char. classes of ν_M have to be).

Ex: $N = \mathbb{R}^n$, so $TN = \underline{\mathbb{R}}^n$.

The Whitney sum formula tells us, for any immersion $M^m \overset{\text{immersion}}{\xrightarrow{e}} \mathbb{R}^n$ (could be an embedding)

since $\underset{\text{rank } m}{TM} \oplus \underset{\text{rank } n-m}{\nu_M} \cong \underline{\mathbb{R}}^n$

$$\Rightarrow \overbrace{w(TM)}^{w=1+w_1+\cdots} \cup \underbrace{w(\nu)}_{\bar{w}=1+\bar{w}_1+\bar{w}_2+\cdots} = 1.$$

← Sometimes called the "Whitney duality formula" (up to stabilizing, ν_M is "dual" or "inverse" via $\oplus$ to TM).

Can solve for $w(\nu)$ as $w(TM)$ is a unit.

in deg 1: $w_1+\bar{w}_1 = 0 \Rightarrow \underline{\bar{w}_1 = -w_1} = w_1$ (mod 2.)

in $\underline{\text{deg } 2}$: $w_2 + w_1\bar{w}_1 + \bar{w}_2 = 0$

$\downarrow$ using deg 1 solution of $\bar{w}_1$

$w_2 + w_1^2 + \bar{w}_2 = 0 \Rightarrow \underline{\bar{w}_2 = w_2 + w_1^2}$.

etc.

For any M, let $\bar{w}(M)$ be the solution to $w(M) \cup \bar{w}(M) = 1$ (know: $w(\nu_M) = \bar{w}(M)$ for any $M \hookrightarrow \mathbb{R}^n$).

e.g., $w(\mathbb{RP}^m) = \underline{(1+h)^{m+1}}$ in $\mathbb{Z}/2[h]/h^{m+1} \cong H^\bullet(\mathbb{RP}^m; \mathbb{Z}/2)$

so $\bar{w}(\mathbb{RP}^m)$ is $\dfrac{1}{(1+h)^{m+1}}$ in $\nearrow$

Let's explicitly compute in some nice cases:

$\underline{\text{identity}}$: $(1+h)^2 = 1+h^2$ over $\mathbb{Z}/2$, similarly $(1+h)^{2^i} = 1+h^{2^i}$ mod 2, so

if $m+1 = \sum n_i 2^i$ ← binary representation of $m+1$,

then over $\mathbb{Z}/2$, $(1+h)^{m+1} = (1+h)^{\sum n_i 2^i} = \prod_{i \text{ s.t. } n_i = 1} (1+h^{2^i})$

e.g., $m=10$: $(\mathbb{RP}^{10})$.

$$\underset{\text{in } \mathbb{Z}/2[h]/h^{11}}{w(\mathbb{RP}^{10}) = (1+h)^{11=1+2+8}} = (1+h)(1+h^2)(1+h^8) = \underset{\text{in } \mathbb{Z}/2[h]/h^{11}.}{1+h+h^2+h^3+h^8+h^9+h^{10}}.$$

To compute $\bar{w}$ in this case (mult. inverse of $(1+h)^{m+1}$ in $\mathbb{Z}/2[h]/h^{m+1}$), observe:

for any s w/ $2^s > m$,

$$\underbrace{(1+h)^{m+1}}_{w}(1+h)^{2^s-(m+1)} = (1+h)^{2^s} \underset{(\text{mod } 2)}{=} 1+h^{2^s} = 1 \quad (h^{2^s} = 0 \text{ in }).$$

$\Rightarrow \bar{w} = (1+h)^{2^s-(m+1)}$ for any such s.

$n=10$ again; e.g.,

$$\bar{w} = (1+h)^{16-(11)} = (1+h)^{5=1+4} = (1+h)(1+h^4) = 1+h+h^4+h^5 ..$$

i.e., $\bar{w}_5 = h^5 \neq 0$. (implies: if $\mathbb{RP}^{10} \looparrowright \mathbb{R}^n$, then $\bar{w}_5(\mathbb{RP}^{10}) = w_5(\nu_M) \neq 0$ so rank $(\nu_M) = n-10 \geq 5$ by dimension reasons).

<u>Cor:</u> If $\mathbb{RP}^{10} \looparrowright \mathbb{R}^n$ then $n \geq 15$.

i.e., $\mathbb{RP}^{10}$ can't immerse or embed into $\mathbb{R}^{14}$.

(we know by whitney embedding any $M^m \hookrightarrow \mathbb{R}^{2m+1}$, but often can do better, e.g., $S^2 \hookrightarrow \mathbb{R}^3$, the above cor puts constraints on how much better one can do for case of $\mathbb{RP}^{10}$).

In general, the amount of "immersion constraint" we'll get for a given $\mathbb{RP}^m$ depends on m. One case in which it's very strong:

$$\mathbb{RP}^{2^k}: \text{ Get } w(\mathbb{RP}^{2^k}) = (1+h)^{2^k+1}, \text{ and } \bar{w}(\mathbb{RP}^{2^k}) = (1+h)^{\overbrace{2^{k+1}}^{2^s \text{ from above (chose } s=k+1)}-2^k-1} = (1+h)^{2^k-1}.$$

$$= \frac{(1+h)^{2^k}}{1+h} = \frac{1+h^{2^k}}{1+h} \underset{\text{mod } 2}{=} \underbrace{1+h+h^2+\cdots+h^{2^k-1}}.$$

Seeing as $\bar{w}_{2^k-1}(T\mathbb{RP}^{2^k}) \neq 0 \Rightarrow$ " The normal bundle of any immersion $\mathbb{RP}^{2^k} \looparrowright \mathbb{R}^n$ $\underbrace{\text{must have dimension}}_{\overset{\shortparallel}{n-2^k}.} \geq 2^k-1$, i.e., $n \geq 2(2^k)-1$.

<u>Cor:</u> For $m=2^k$, $\mathbb{RP}^m$ can't immerse (hence can't embed either) into $\mathbb{R}^{2m-2}$

(Whitney's immersion theorem states any $M^m \looparrowright \mathbb{R}^{2m-1}$, & Cor. states that for $\mathbb{RP}^m$, $m=2^k$ we can't $\looparrowright$ into anything lower).

<u>Stiefel-Whitney numbers</u>

X^n compact smooth manifold, . $w_i(X) := w_i(TX) = w_i \in H^i(X;\mathbb{Z}/2)$,

Can multiply $\prod w_i(X)^{n_i} \in H^{\sum i n_i}(X;\mathbb{Z}/2)$.

Recall that $\exists$ a canonical $\mathbb{Z}/2$ fundamental class $[X] \in H_n(X;\mathbb{Z}/2)$ (don't need orientability mod 2) determining a morphism $H^n(X;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$ (iso. if X connected)

$$\phi \longmapsto \langle \phi, [X] \rangle, \text{ or } \phi([X]).$$

$\Rightarrow$ Whenever $\sum i n_i = n$ we get a number, denoted $\prod w_i^{n_i}[X] \in \mathbb{Z}/2$ by $\langle \prod w_i(X)^{n_i}, [X] \rangle$.

Stiefel-Whitney # of X.

0 in $\mathbb{Z}_2[h]/h^5$.

E.g., if $X = \mathbb{R}P^4$, $w(X) = (1+h)^{5=1+4} = (1+h)(1+h^4) = 1 + \underset{w_1}{h} + \underset{w_4}{h^4} + \cancel{h^5}$

the possible numbers here are:

$w_1^4[X]$

$w_2 w_1^2 [X]$

$w_2^2 [X]$

$\vdots$

$w_4[X]$

but not all are non-zero, i.e., $w_2(\mathbb{R}P^4) = 0$ so $w_2 w_1^2 [\mathbb{R}P^4] = 0$, & e.g., $w_4[X] = 1$.

e.g., a cobordism from $\emptyset$ to X is a W w/ $\partial W = X$.

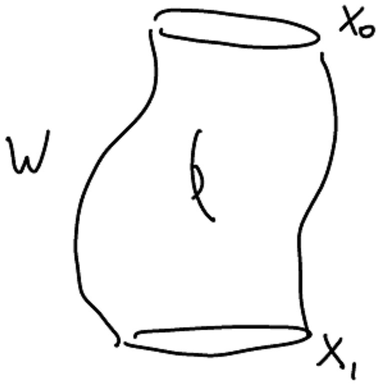
Cobordism: X_0, X_1 compact smooth manifolds. A cobordism W between X_0 & X_1 is a smooth compact manifold with boundary W s.t.

$$\partial W = X_1 \sqcup X_0;$$

(we'll suppress α from arguments).

more precisely allow pairs (W, α) W m'fld w/ ∂ (compact, smooth), & $\alpha : \partial W \xrightarrow[\text{diffeo.}]{\cong} X_0 \sqcup X_1$

(in particular, if X_0, X_1 are diffeo, they're cobordant).



W is a cobordism between $X_0 = S^1$ and $X_1 = S^1$.

We say X_0, X_1 are cobordant if $\exists$ a cobordism between them.

(this is an equivalence relation: X cobordant to X always via $W = X \times I$;

& e.g., if X cobordant to Y via W_{XY}, Y cob. to Z via W_{YZ}, then

X cobordant to Z via $W_{XY} \cup_Y W_{YZ}$.)

exercise: check can be made a smooth manifold (using $\exists$ of smooth collar nbhds near boundary)

In particular ($X_0 = \emptyset$), if $\overset{\text{cpct m'fold}}{X} = \partial \overset{\text{cpct w/ } \partial}{W}$ then all Stiefel-Whitney #'s of X are 0.

Prop: If X_0 & X_1 are cobordant they have the same Stiefel-Whitney #'s.

(Thom proved converse is true also, but that's much harder)

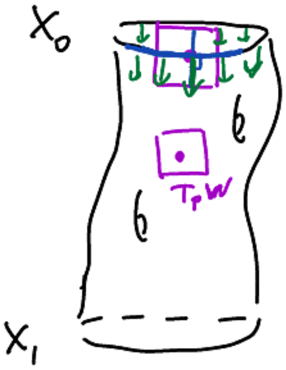
Cor: $\mathbb{R}P^4$ and S^4 are not cobordant. (we computed above that $w_4[\mathbb{R}P^4] = 1$ but $w_4[S^4] = 0$ b/c $w_4(S^4) = 0$ by last class).

special case of above

Cor: $\mathbb{R}P^4$ is not ∂W for a cpct m'fold-w/-∂ W.

Pf: Write $w = \prod w_i^{n_i}$ where $\sum i n_i = n = \dim X_i$. $i = 0, 1$. $\left(w(E) = \prod w_i^{n_i}(E). \right)$

The basic observation is that for a cobordism W:



A choice of inward pointing vector field along $TW|_{X_i}$ (always exists by partition of unit argument – Math 535a) leads to a decomposition $\underline{TW|_{X_i} \cong TX_i \oplus \underline{\mathbb{R}}}$

Therefore

$$w(TW)\Big|_{X_i} = w(TW|_{X_i}) = w(TX_i \oplus \underline{\mathbb{R}}) \underset{\text{Whitney sum}}{=} w(TX_i)$$

means pull back along $X_i \hookrightarrow W$

Furthermore, if $[X_i] \in H_n(X_i; \mathbb{Z}/2)$ denotes the canonical $\mathbb{Z}/2$ fund. classes, and $[W] \in H_{n+1}(W, \partial W; \mathbb{Z}/2)$ denotes the canonical rel. $\mathbb{Z}/2$ fund. class of W. we know (or at least previously asserted) that in LES of pair $(W, \partial W)$ w/ $\mathbb{Z}/2$-coeffs,

$$\begin{tikzcd} H_{n+1}(W, \partial W) \arrow[r, "\partial_*"] & H_n(\partial W) \arrow[r, "i_*"] & H_n(W) \\ {[W]} \arrow[r, mapsto] & {[\partial W]} & \end{tikzcd} \qquad \partial_* : [W] \longmapsto [\partial W].$$

Cor: if $i : \partial W \to W$ then $i_*[\partial W] = 0$ in $H_n(W)$.

$$\begin{array}{c} i_*[\partial W] \\ \| \\ i_*[X_0] + i_*[X_1] \end{array} \quad (\text{as } \partial W = X_0 \amalg X_1).$$

$$\Rightarrow (\text{mod } 2) \quad i_*[X_0] = i_*[X_1]$$

Therefore if $w = \prod w_i^{n_i}$

$$\Rightarrow \langle w(TW), i_*[X_0] \rangle = \langle w(TW), i_*[X_1] \rangle \overset{\text{naturality}}{=\!=} \langle w(TW|_{X_1}), [X_1] \rangle$$

$$\begin{array}{l} \langle w(TW), i_*[X_0] \rangle \\ \| \text{ by naturality} \\ \langle w(TW|_{X_0}), [X_0] \rangle \\ \| \text{ before} \\ \langle w(TX_0), [X_0] \rangle \\ \| \\ \prod w_i^{n_i}[X_0]. \end{array} \qquad \begin{array}{l} \langle w(TW|_{X_1}), [X_1] \rangle \\ \| \text{ before} \\ \langle w(TX_1), [X_1] \rangle \\ \| \\ \prod w_i^{n_i}[X_1] \end{array}$$

☒.

3/26/2021

Some computations of Chern classes and Chern numbers.

One source of complex vector bundles comes from the tangent bundle to a complex manifold, as we'll now explain.

Some (fiber-wise) linear algebra.

V real vec. space of dim. $2n$. A cplx str. on V is $J: V \to V$ w/ $J^2 = -id$.

Using J, V inherits str. of a $\mathbb{C}$-n-dim'l vec. space via $(a+bi)(v) := (a+bJ)(v)$, call this cplx. vector space (V, J).

Given a real vector bundle $E \to X$ of rank $2n$, a $J \in \Gamma(End(E))$ (i.e., $J_x: E_x \to E_x$) w/ $J^2 = -id$ (meaning $J_x^2 = -id$ for all x) induces a complex vec. bundle str. on E, call it (E, J).
and hence chern classes

Call such a J a (fiberwise) complex structure on E.

Call a pair (X, J) an <u>almost complex manifold</u> & such a J on TX an <u>almost complex structure</u>.

manifold

fiberwise complex structure on TX

An almost cplx manifold (X, J) has $c_j(X) := c_j(TX, J)$.

A <u>complex manifold</u> (dim. n) is a space equipped w/ (equiv. class of, or maximal) atlas $\{\underset{X}{\underset{\cap}{U_\alpha}}, \phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \underset{open}{\subset} \mathbb{C}^n\}_\alpha$,

whose transition functions

$$\phi_\beta \circ \phi_\alpha^{-1} : \underset{\mathbb{C}^n}{\underset{\cap \text{ open}}{\phi_\alpha(U_\alpha)}} \longrightarrow \underset{\mathbb{C}^n}{\underset{\cap \text{ open}}{\phi_\beta(U_\beta)}}$$

are <u>holomorphic</u>,

meaning that $d(\phi_\beta \circ \phi_\alpha^{-1}) \circ i = i \circ d(\phi_\beta \circ \phi_\alpha^{-1})$.

<u>Lem</u>: Any complex manifold X has a canonical almost complex structure, hence TX is a cplx. vec. bdle (& has Chern classes).

<u>Sketch</u>: At a given $p \in X$, pick a chart U_α around p, giving

$$T_p X \xrightarrow[(d\phi_\alpha)_p]{\cong} T_{\phi_\alpha(p)}(\phi_\alpha(U_\alpha)) \cong \mathbb{C}^n \circlearrowleft i$$

Define J_p to be $(d\alpha)_p^{-1}\, i\, (d\phi_\alpha)_p$; check independent of choice & smoothly varying. (uses holomorphicity of transition functions). $\boxtimes$.

<u>Ex</u>: $G_k(\mathbb{C}^n)$. We can construct a complex differentiable atlas parallel to the (real) atlas we constructed for $G_k(\mathbb{R}^n)$.

(i.e., around $E_0 \in G_k(\mathbb{C}^n)$, obtain (inverse to) a chart map by

using the standard Hermitian inner prod. on $\mathbb{C}^n$.

$$\Psi : \underset{\cong\, \mathbb{C}^{k(n-k)}}{\mathrm{Hom}_{\mathbb{C}}(E_0, E_0^{\perp})} \longrightarrow G_k(\mathbb{C}^n)$$

$$a \longmapsto \mathrm{graph}(a)(E_0).\quad)$$

$$\underset{\underset{(id,\,a)}{\shortparallel}}{\mathrm{graph}(a)} : E_0 \hookrightarrow E_0 \oplus E_0^{\perp} \cong \mathbb{C}^n$$

(exercise: complex manifold)

The same analysis previously applied to $G_k(\mathbb{R}^n)$ implies that as complex vector bundles

$$T\,Gr_k(\mathbb{C}^n) \cong \underline{\mathrm{Hom}}_{\mathbb{C}}(E, E^{\perp})$$

E: tautological bundle over $G_k(\mathbb{C}^n)$; $E^{\perp}$: inside $\underline{\mathbb{C}}^n$ using $\langle -,- \rangle_{\text{Euclidean}}$ Hermitian metric.

$\underline{k=1}$ $\quad (G_1(\mathbb{C}^n) \cong \mathbb{C}P^{n-1})$

$$\Rightarrow T\mathbb{C}P^{n-1} \oplus \underline{\mathbb{C}} \underset{\text{as before}}{\cong} \underline{\mathrm{Hom}}_{\mathbb{C}}(L_{taut}, L_{taut}^{\perp}) \oplus \underset{\underset{\underset{\underline{\mathbb{C}}}{\cong}}{L_{taut}^{*} \otimes L_{taut}}}{\overset{}{\mathrm{Hom}(L_{taut}, L_{taut})}}$$

$$= \underline{\mathrm{Hom}}(L_{taut}, \underline{\mathbb{C}}^n) = \underbrace{L_{taut}^{*} \oplus \cdots \oplus L_{taut}^{*}}_{n \text{ times}}$$

Now for any cplx line bundle $L \to B$, $\underline{c_1(L^*) = -c_1(L)}$.

$$\Big(b/c\ \ \underset{\overset{\shortparallel}{c_1(L)+c_1(L^*)}}{c_1(L \otimes L^*)} = c_1(\underline{\mathbb{C}}) = 0\Big).$$

h canonical generator in $H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$.

So, $c_1(L_{taut}^{*}) = -c_1(L_{taut}) = -(-h) = h$.

$$\text{so, } c(\mathbb{C}P^{n-1}) := c(T\mathbb{C}P^{n-1}) \underset{\text{whitney sum}}{=} c\big((L_{taut}^{*})^{\oplus n}\big) \underset{\text{whitney sum}}{=} \prod_{i=1}^{n} \overbrace{c(L_{taut}^{*})}^{(1+h)}$$

$$= (1+h)^n \quad \text{in } \underset{\underset{\mathbb{Z}[h]/h^n.}{\cong}}{H^{*}(\mathbb{C}P^{n-1}; \mathbb{Z})}$$

$$1 + nh + \binom{n}{2}h^2 + \cdots + nh^{n-1}.$$

$$\text{i.e., } c_i(\mathbb{C}P^{n-1}) = \binom{n}{i} h^i \in \underbrace{H^{2i}(\mathbb{C}P^{n-1}; \mathbb{Z})}_{\underset{\mathbb{Z}\langle h^i \rangle.}{\cong}}.$$

Above it was convenient to know relationship between c_1's for L, L^*. What about E vs. E^*? (& all c_i's)

Lemma: E rank k complex vector bundle, and let $E^* := \underline{\mathrm{Hom}}_{\mathbb{C}}(E, \mathbb{C})$.

Then for each i, $c_i(E^*) \cong (-1)^i c_i(E)$.

Pf: • true when $\mathrm{rank}(E) = 1$, by above. ($c_1(L^*) = -c_1(L)$, $c_i(L^*) = 0 = (-1)^i c_i(L)$ for $i > 1$, & $c_0(L^*) = 1 = c_0(L)$).

• true when $E \cong L_1 \oplus \cdots \oplus L_k$. $(\Rightarrow E^* \cong L_1^* \oplus \cdots \oplus L_k^*)$.

$$\Rightarrow \quad c(E^*) = \prod_{i=1}^{k} c(L_i^*) = \prod_{i=1}^{k} (1 - c_1(L_i))$$

$$\text{vs. } c(E) = \prod_{i=1}^{k} (1 + c_1(L_i))$$

; now check in deg $2i$: these differ by $(-1)^i$.

• In general, by splitting principle, $\exists \; s : Z \to \overset{\substack{E \\ \downarrow}}{X}$ w/ $s^*E \cong L_1 \oplus \cdots \oplus L_k$.

So it follows from previous case that

$$\begin{tikzcd}
(-1)^i c_i(s^*E) \arrow[r, phantom, "="] \arrow[d, phantom, "\parallel"] & c_i((s^*E)^*) \arrow[d, phantom, "\parallel"] \\
s^*((-1)^i c_i(E)) \arrow[r, dashed, no head, "="] & s^*(c_i(E^*))
\end{tikzcd}$$

s^* is injective so $(-1)^i c_i(E) = c_i(E^*)$ ✓. $\boxtimes$.

V vector space $/ \mathbb{C}$, have $\overline{(-)} : \mathbb{C} \xrightarrow{\cong} \mathbb{C}$ real-linear involution; pulling back the action
$a + ib \longmapsto a - ib$

of $\mathbb{C}$ on V by $\overline{(-)}$ gives a new complex vector space $\overline{V}$; as real vector spaces $V_{\mathbb{R}} = (\overline{V})_{\mathbb{R}}$,

but

$$(a+bi) \cdot v := (a - ib) \cdot v$$

scalar mult. in $\overline{V}$ — scalar mult. in V.

Observe that a Hermitian inner product $\langle -, - \rangle$ on V induces an isomorphism $V^* \overset{\mathbb{C}-}{\cong} \overline{V}$.

complex linear in this factor — complex antilinear in this factor, meaning linear when thought of as a map from $\overline{V}$.

Similarly, from a complex vector bundle $\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$, can construct $\begin{smallmatrix} \overline{E} \\ \downarrow \\ X \end{smallmatrix}$, & a choice of (fiberwise) Hermitian metric gives an iso. $\overline{E} \cong E^*$.

Cor: $c_i(\overline{E}) = (-1)^i c_i(E)$.

We've studied the effect of $\oplus$, $(-)^*$, $\otimes$ char. classes, but $\otimes$ only for line bundles so far.

What about $\otimes$ for other vector bundles? In general, there's not a clean formula; however can use splitting principle to deduce formulae in each degree in any given example.

Ex: $G := Gr_2(\mathbb{C}^4)$, calculate $c_i(G) := c_i(TG)$ in terms of $c_1(\overbrace{E_{taut}}^{E})$, $c_2(E_{taut})$

(using $TG \cong \underline{Hom}_{\mathbb{C}}(E_{taut}, E_{taut}^{\perp}) \cong E_{taut}^{*} \otimes (E_{taut}^{\perp})$.)

First note that $TG = \underline{Hom}_{\mathbb{C}}(E, E^{\perp})$ so $TG \oplus Hom(E,E) \cong Hom_{\mathbb{C}}(E, E^{\perp} \oplus E)$

$$\cong Hom_{\mathbb{C}}(E, \underline{\mathbb{C}}^4)$$

$$\boxed{TG \oplus Hom(E,E) = (E^*)^{\oplus 4}.}$$

$c_i(E^*) = (-1)^i c_i(E)$

so, need to compute c of this bundle.

Let's assume can find L_1, L_2 with $E \cong L_1 \oplus L_2$. (actually we cannot, but by splitting principle, we can pull back to such a Z where splitting holds, derive identities which remain true on original manifold) in that case, denoting by $l_i := c_1(L_i)$, $E^* \cong L_1 \oplus L_2$

$$c(E) = (1+l_1)(1+l_2) = 1 + \underbrace{(l_1+l_2)}_{c_1(E)} + \underbrace{l_1 l_2}_{c_2(E)}$$

$$c(E^*) = (1-l_1)(1-l_2)$$

$$c((E^*)^{\oplus 4}) = (1-l_1)^4(1-l_2)^4.$$

$$c(hom(E,E)) = c(E^* \otimes E) = c\left(\bigoplus_{i,j=1}^{2} L_i^* \otimes L_j\right)$$

$$c(L_i^* \otimes L_j) = 1 + l_j - l_i$$

$$= \prod_{\substack{i,j=1 \\ i \neq j}}^{2} (1 + l_j - l_i) = (1 + l_1 - l_2)(1 + l_2 - l_1)$$

$$\text{or } 1 - (l_1 - l_2)^2$$

The Whitney sum formula now implies that

$$\underset{(1+c_1(TG)+c_2(TG)+c_3(TG)+c_4(TG))}{c(TG)} \cup \left(\prod_{i \neq j} 1 + l_j - l_i\right) = (1-l_1)^4(1-l_2)^4.$$

Using this equation, can solve for $c_i(G)$:

$$c_1(G) = -4(l_1+l_2) = -4c_1(E)$$

$$c_2(G) = 7c_1(E)^2 = 7(l_1+l_2)^2$$

$$c_3(G) = -6c_1(E)^3$$

$$c_4(G) = 3c_1(E)^4 - 4c_1(E)^2 c_2(E) + 4c_2(E)^2.$$

One can use this to calculate Chern #'s of G, at least in terms of integrals of chern classes of $E_{(taut)}$.

e.g., $c_2^2[G] = \langle 49 c_1(E)^4, [G] \rangle$.

$c_2^2(G) \in H^8(G; \mathbb{Z})$

G is real 8.-dim'l / cplx 4 dimensional.