Linear algebra of complexifications

V real vec. space dim n.

$\rightsquigarrow$

$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ complexification.. complex vec. space of dim n.

observe: in contrast to arbitrary complex vec. space, V comes equipped w/ a canonical conjugation action:

$\mathbb{C} \xrightarrow{\overline{(-)}} \mathbb{C}$ induces (by $V \otimes_{\mathbb{R}} -$) $V_{\mathbb{C}} \xrightarrow[\cong]{\overline{(-)}} V_{\mathbb{C}}$ complex anti-linear isomorphism, i.e., induces a complex-linear isomorphism $V_{\mathbb{C}} \xrightarrow{\cong} \overline{V_{\mathbb{C}}}$.

Can recover V as $\mathrm{Fix}(V_{\mathbb{C}} \circlearrowleft \overline{(-)})$ (i.e., $+1$-eigenspace: note $\overline{(-)}^2 = \mathrm{id}$)

If W is a complex vector space ($\dim_{\mathbb{C}} = n$), denote by $W_{\mathbb{R}}$ the underlying real vector space ($\dim_{\mathbb{R}} 2n$).

scalar mult. in W

Multiplication by i on W $\longleftrightarrow$ $J : W_{\mathbb{R}} \to W_{\mathbb{R}}$, $w \mapsto iw$, w/ $J^2 = -\mathrm{id}$ ("complex structure on $W_{\mathbb{R}}$")

Lem: If W ($\dim_{\mathbb{C}} n$) complex vector space then $\underset{\dim_{\mathbb{R}} 2n}{(W_{\mathbb{R}})_{\mathbb{C}}} := \overbrace{(W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})}^{\dim_{\mathbb{C}} 2n} \cong W \oplus \overline{W}$.

Pf sketch: mult. by i on W induces as above $J : W_{\mathbb{R}} \to W_{\mathbb{R}}$ w/ $J^2 = -\mathrm{id}$.

$\Rightarrow$ get $J_{\mathbb{C}} = J \otimes_{\mathbb{R}} \mathrm{id}_{\mathbb{C}} : (W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \to (W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$ w/ $(J_{\mathbb{C}})^2 = -\mathrm{id}$,

i.e., $J_{\mathbb{C}}$ has $(+i)$ and $(-i)$ eigenspaces, which together give a decomposition $(W_{\mathbb{R}})_{\mathbb{C}}$.

i.e., $W_{\mathbb{R}} \otimes \mathbb{C} \cong W^+ \oplus W^-$ $\qquad W^{\pm} := \pm i$ eigenspace.

as $\mathbb{C}$-vec. spaces

So need to show $W^+ \cong W$ (& $W^- \cong \overline{W}$, & $\overline{(-)}$ on $(W_{\mathbb{R}})_{\mathbb{C}}$ swaps W & $\overline{W}$ factors).

– exercise

Define $W \xrightarrow{T} W^+$; on the level of real vector spaces $W_{\mathbb{R}} \xrightarrow{v \mapsto v \otimes 1} (W_{\mathbb{R}})_{\mathbb{C}} \xrightarrow{pr_{+i}} W^+$

which all together sends $w \xmapsto{T} \frac{1}{2}(w \otimes 1 - Jw \otimes i)$. $\qquad \alpha \longmapsto \frac{1}{2}(\alpha - iJ_{\mathbb{C}}\alpha)$

check: $Jw \xmapsto{T} i(Tw)$; in particular.

$\alpha = (\alpha^+, \alpha^-) \in W^+ \oplus W^-$,

$iJ\alpha = (-\alpha^+, +\alpha^-)$

$\frac{1}{2}(\alpha - iJ\alpha) = (\alpha^+, 0)$.

T is a complex-linear map $W \to W^+$, isomorphism (check).

∎.

Pontryagin classes of real vector bundles

$E \to X$ real vec. bundle of rank k.

Form $E \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow X$ (fiberwise) complexification, complex rank k vec. bundle w/ an iso.

$$E \otimes_{\mathbb{R}} \mathbb{C} \underset{\text{conjugation}}{\cong} \overline{E \otimes_{\mathbb{R}} \mathbb{C}} \underset{\substack{\text{using}\\ \text{Hermitian metric}\\ \text{as in last time}}}{\cong} (E \otimes_{\mathbb{R}} \mathbb{C})^* . \quad (\star)$$

Taking Chern classes $c_i(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{2i}(X;\mathbb{Z})$, and $(\star)$ implies:

$$\underline{c_i(E \otimes_{\mathbb{R}} \mathbb{C})} = c_i((E \otimes_{\mathbb{R}} \mathbb{C})^*) \underset{\text{last time}}{=} (-1)^i \, \underline{c_i(E \otimes_{\mathbb{R}} \mathbb{C})}.$$

If i is odd, this tells us that $2c_i(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$ in $H^{4k+2}(X;\mathbb{Z})$.
$i = 2k+1$

Def: $E \to X$ real vec. bundle of rank k, define its k^{th} Pontryagin class by

$$p_k(E) := (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}(X;\mathbb{Z}).$$

By definition, $p_k(E) = 0$ if $2k > \operatorname{rank}(E)$.

Whitney sum formula: E, E' two vector bundles, then

$$p_k(E \oplus E') := (-1)^k c_{2k}((E \otimes_{\mathbb{R}} \mathbb{C}) \oplus (E' \otimes_{\mathbb{R}} \mathbb{C}))$$

$$\underset{\substack{\text{(Whitney sum for}\\ \text{Chern classes)}}}{=} (-1)^k \underbrace{\sum_{\substack{i+j=2k\\ i \geq 0\\ j \geq 0}} c_i(E \otimes_{\mathbb{R}} \mathbb{C}) \smile c_j(E' \otimes \mathbb{C})} \qquad (\text{convention } c_0 = 1)$$

terms where both i, j even

terms where one of i or j is odd.

$$= \sum_{r+s=k} (-1)^{k=r+s} c_{2r}(E \otimes \mathbb{C}) \smile c_{2s}(E \otimes \mathbb{C}) + (2\text{-torsion terms})$$

$$= \sum_{\substack{r+s=k\\ r \geq 0,\ s \geq 0}} p_r(E) \smile p_s(E') + (2\text{-torsion terms}).$$

$\longleftarrow$ convention that $p_0 = 1$.

So denoting $p(E) := \underbrace{1}_{\substack{\| \\ p_0(E)}} + p_1(E) + p_2(E) + \cdots$ total Pontryagin class,

get
$$p(E \oplus E') = p(E)p(E') + 2\text{-torsion terms}.$$

Special case:

Say $E = F_{\mathbb{R}}$ rank $2n$ real vec. bundle for $F \to X$ a complex rank n vec. bundle.

Then, a fiberwise version of the lemma at the start of lecture implies:
$$(F_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \cong F \oplus \overline{F} \underset{\substack{\text{Hermitian} \\ \text{metric on } F}}{\cong} F \oplus F^*.$$

So, $\boxed{p_k(F_{\mathbb{R}})} = (-1)^k c_{2k}(F_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^k c_{2k}(F \oplus F^*)$

$$\underset{\substack{\text{Whitney sum} \\ \text{for Chern classes}}}{=} (-1)^k \sum_{\substack{i+j=2k \\ i\geq 0\ j\geq 0}} c_i(F) \cup c_j(F^*) = \boxed{(-1)^k \sum_{\substack{i+j=2k \\ i\geq 0\ j\geq 0}} (-1)^j c_i(F) \cup c_j(F)}$$

As usual if Q a (real) smooth manifold denote $p_k(Q) := p_k(TQ)$.

Example: Compute $p_k(\mathbb{CP}^n)$.

We previously computed as complex vector bundles, $T\mathbb{CP}^n \oplus \underline{\mathbb{C}} \cong \underbrace{L^* \oplus \cdots \oplus L^*}_{n+1 \text{ copies}}$ (★) $L :=$ taut. line bundle

$$\Rightarrow c(T\mathbb{CP}^n) = (\underbrace{1+h}_{c(L^*)})^{n+1} \quad \text{in } H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}.$$

complex conjugating (★), we get:
$$\overline{T\mathbb{CP}^n \oplus \underline{\mathbb{C}}} \cong \overline{\underbrace{L^* \oplus \cdots \oplus L^*}_{n+1}} \cong \underbrace{L \oplus \cdots \oplus L}_{n+1}$$
$$\| $$
$$\overline{T\mathbb{CP}^n} \oplus \underline{\mathbb{C}} \quad (\overline{\mathbb{C}} \cong \mathbb{C}).$$

$$\Rightarrow c(\overline{T\mathbb{CP}^n}) = c(L)^{n+1} = (1-h)^{n+1}, \text{ in same ring}$$

So, $p_k(\mathbb{CP}^n) = p_k(T\mathbb{CP}^n) = (-1)^k c_{2k}(T\mathbb{CP}^n \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^k c_{2k}(T\mathbb{CP}^n \oplus \overline{T\mathbb{CP}^n})$.

$$= (-1)^k \cdot \left(\deg 2k \text{ part of } (1+h)^{n+1}(1-h)^{n+1}\right).$$

So, $p(\mathbb{C}P^n) = \sum_{k\geq 0} (-1)^k \left((1+h)^{n+1}(1-h)^{n+1} \right)_{\text{deg } 2k \text{ part}}$

$$= \sum_{k\geq 0} (-1)^k \Big(\underbrace{(1-h^2)^{n+1}}_{} \Big)_{\text{deg } 2k \text{ part}}$$

deg $2k$ part is $(-1)^k \cdot$ deg $2k$ part of $(1+h^2)^{n+1}$, & no odd degree parts of this expression, hence.

$$\boxed{= (1+h^2)^{n+1}} \qquad (\text{again in } H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}).$$

Special case: $n = 2m$ is even. We get $p(\mathbb{C}P^{2m}) = (1+h^2)^{2m+1}$.

In particular $p_m(\mathbb{C}P^{2m}) := p_m(T\mathbb{C}P^{2m}) = \binom{2m+1}{m} h^{2m} \in H^{4m}(\mathbb{C}P^{2m};\mathbb{Z}) \cong \mathbb{Z}\langle h^{2m}\rangle$

↑ top degree cohomology ($\dim_{\mathbb{R}} \mathbb{C}P^{2m} = 4m$).

Pairing w/ the fundamental class $[\mathbb{C}P^{2m}]$ (using complex orientation) sends $h^{2m} \longmapsto +1$, hence we get

$$\underbrace{\langle p_m(\mathbb{C}P^{2m}), [\mathbb{C}P^{2m}] \rangle}_{\text{the Pontryagin number } p_m[\mathbb{C}P^{2m}].} = \binom{2m+1}{m}$$

More generally, if X compact oriented manifold, for any collection $\{n_i \geq 0\}$ with $\sum 4i\, n_i = \dim X$, can define

$$\prod p_i^{n_i}[X] := \langle \underbrace{\prod p_i(TX)^{n_i}}_{\in H^{\dim X}(X;\mathbb{Z}) \text{ by hypothesis}}, [X] \rangle \in \mathbb{Z}.$$

Pontryagin numbers.

Observe: If $X \xrightarrow[f]{\cong} Y$ oriented diffeo. (so $f_*[X] = [Y]$) then naturality $\Rightarrow$

$\prod p_i^{n_i}[X] = \prod p_i^{n_i}[Y]$.

On the other hand, $\prod p_i^{n_i}[\overline{X}] = -\prod p_i^{n_i}[X]$.

↑ means X w/ opposite orientation, $-[X]$.

Cor: If a single Pontryagin # is non-zero, then $X \underset{\text{oriented diff.}}{\not\cong} \overline{X}$.

Cor: $\mathbb{C}P^{2m} \not\cong \overline{\mathbb{C}P^{2m}}$.

Interestingly enough, $\mathbb{C}P^{2m+1} \cong \overline{\mathbb{C}P^{2m+1}}$, e.g., $\mathbb{C}P^1 = S^2 \xrightarrow{\text{reflection}} S^2 = \mathbb{C}P^1$.

oriented

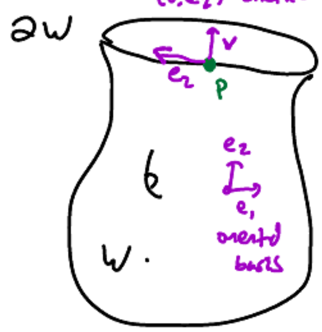
Oriented cobordism:

Now we'll consider W^{n+1} := compact smooth $(n+1)$-dim'l manifold w/ boundary, equipped w/ an orientation

$\rightsquigarrow$ get $[W] \in H_{n+1}(W, \partial W; \mathbb{Z})$.

If such a W is orientable (which we're assuming), then ∂W is too, & an orientation on W determines one on ∂W: the convention we'll use is "outward normal first":

(v, e_2) oriented basis in T_pW, so e_2 is an oriented basis for $T_p \partial W$.



If $p \in \partial W$, $v \in T_pW$ any 'outward' pointing tangent vector, then we declare $(e_2, \dots, e_n) \in T_p\partial W$ to be positively oriented iff $(v, e_2, \dots, e_n)$ is positively oriented basis in T_pW.

$\rightsquigarrow$ get using this convention, a class $[\partial W] \in H_n(\partial W; \mathbb{Z})$.

This convention is compatible w/ connecting homomorphism:

Lem: The map $\partial_* : H_{n+1}(W, \partial W; \mathbb{Z}) \longrightarrow H_n(\partial W; \mathbb{Z})$ sends $[W] \longmapsto [\partial W]$.

(omitted).

We'll sometimes denote an oriented manifold by $X := (X, \omega)$ & opposite orientation by $\overline{X} = (X, -\omega)$

choice of orientation

(section of $\tilde{X} \to X$ if exists or $\Leftrightarrow$ a section of $\mathrm{Frame}(TX) \times_{GL(n,\mathbb{R})} \mathbb{Z}/2$) where $\mathbb{Z}/2 := GL(n)/GL(n)^+$

inherits an action of $GL(n)$ by "sign of det").

oriented cpct m'fld-with-∂.

Say W is an <u>oriented cobordism</u> from $X_0 = (X_0, \omega_0)$ to $X_1 = (X_1, \omega_1)$ is $\partial W \underset{\text{oriented diff}}{=} \overline{X}_0 \amalg X_1$.

oriented manifold

<u>Example</u>: $W := X \times [0,1]$ is an oriented cobordism from X to X.

- Any W w/ $\partial W = X$ as oriented manifolds can be thought of as an oriented cob. from $\emptyset$ to X.

Given such a W, if $i := \overline{X}_0 \amalg X_1 = \partial W \hookrightarrow W$; LES of $(W, \partial W)$

$\Rightarrow$ since $\partial_*[W] = [\partial W]$, then $i_*[\partial W] = 0$ in $H_n(W)$

$\qquad = i_*([X_1]) - i_*[X_0]$.

$\Rightarrow i_*[X_0] = i_*[X_1]$ in $H_n(W;\mathbb{Z})$.

Using this, as before (for Stiefel-Whitney #'s) we get:

Thm: Pontryagin #'s are invariant under oriented cobordism.

$\Leftrightarrow$ If $X = \partial W$ as oriented m'flds, then all Pontryagin #'s of X are 0.

Cor: $\underbrace{\mathbb{CP}^{2m}}_{\text{real dim}_{\mathbb{R}} = 4m}$ is not the oriented boundary of any cpct oriented $(4m+1)$-dim'l m'fold.

(note in contrast that $\mathbb{CP}^1 = S^2 = \partial B^3$).

Also similar cor for $\coprod_r \mathbb{CP}^{2m}$, w/ same orientation for each copy.

(of course $\mathbb{CP}^{2m} \amalg \overline{\mathbb{CP}^{2m}}$ is $\partial(\mathbb{CP}^{2m} \times [0,1])$).

3/31/2021

Today: want to compute the cohomology of $BU(k)$ resp. $BO(k)$. (why? any char. class of cplx. resp. real vector bundles of rank k is pulled back from a coh. class in $BU(k)$ resp. $BO(k)$ via classifying map, hence the computation would tell us what all possible such char. classes could be).

$BU(k) = G_k(\mathbb{C}^\infty)$, $BO(k) = G_k(\mathbb{R}^\infty)$

We'll focus on $BU(k) = G_k(\mathbb{C}^\infty)$ ($BO(k)$ case, as usual is parallel provided we work w/ $\mathbb{Z}_2$ instead of $\mathbb{Z}$-coeffs.)

To analyze space, start w/

$$\begin{array}{c} E_{taut} \\ \downarrow \\ G_k(\mathbb{C}^\infty) \end{array}$$

The idea will be to use some form of splitting principle (a particular splitting map $s: Z \to BU(k)$) to embed $H^*(G_k(\mathbb{C}^\infty))$ into H^*(simpler space which can be computed)

The usual proof of the splitting principle produces a space $Z = \mathbb{F}(E_{taut})$ [fiberwise flags in E_{taut}] $= \mathbb{F}_k(\mathbb{C}^\infty)$. One option would be to use this space & compute $H^*(\mathbb{F}(E_{taut}))$ explicitly by making use of Leray-Hirsch applied to various fibrations

e.g.,
$$\begin{array}{ccc} \mathbb{F}_k(\mathbb{C}^\infty) & \longrightarrow & \mathbb{CP}^\infty \\ (L_1,\dots,L_k) & \longmapsto & L_1 \end{array}$$
w/ fiber $\mathbb{F}_{k-1}$ ... see Hatcher's Alg. Topology book §4.

We'll take a shortcut by appealing to a different (somewhat simpler) splitting map (c.f. [Husemoller, Fibre Bundles]).

Consider: $X = \underbrace{\mathbb{CP}^\infty \times \dots \times \mathbb{CP}^\infty}_{k\text{-times}}$ On X we have the rank k vector bundle $E = L_{taut} \times \dots \times L_{taut}$.

Equivalently, $E := \bigoplus_{i=1}^k \pi_i^* L_{taut}$, $\pi_i : X \to \mathbb{CP}^\infty$ proj. to i^{th} factor.

Since $BU(k)$ classifies rank k vector bundles, $\exists!$ (up to homotopy)

$$f_k : X \longrightarrow BU(k) \text{ with } \underbrace{f_k^* E_{taut} = E := \bigoplus_{i=1}^k \pi_i^* L_{taut}}.$$

Prop: f_k is a splitting map for E_{taut}, i.e., ($f_k^* E_{taut}$ splits into line bundles) and f_k^* is injective.

Pf: Let $s : Z \to BU(k)$ be any splitting map for E_{taut} ($\exists$ by splitting principle), i.e.,

$s^* E_{taut} = L_1 \oplus \cdots \oplus L_k$ for $L_i \to Z$ and s^* is injective.

Since each L_i is a complex line bundle, it is classified by a map $g_i : Z \longrightarrow \mathbb{CP}^\infty$ (so $g_i^* L_{taut} = L_i$). Now consider $g = (g_1, \ldots, g_k) : Z \longrightarrow (\mathbb{CP}^\infty)^k$, and let's observe that $g^*(E = \bigoplus_{i=1}^k \pi_i^* L_{taut}) = \bigoplus_{i=1}^k g^* \pi_i^* L_{taut} = \bigoplus_{i=1}^k g_i^* L_{taut} = \bigoplus_{i=1}^k L_i = s^* E_{taut}$.

In particular, $f_k \circ g : Z \to (\mathbb{CP}^\infty)^k \to BU(k)$ classifies $s^*(E_{taut})$, because

$$(f_k \circ g)^*(E_{taut}) = g^* f_k^* E_{taut} = g^* E = s^* E_{taut}.$$

But $s : Z \longrightarrow BU(k)$ classifies $s^* E_{taut}$ by definition. Since classifying maps are unique up to homotopy,

$\Rightarrow f_k \circ g \simeq s$.

$\Rightarrow s^* = g^* f_k^*$. But s^* is injective. $\Rightarrow f_k^*$ is injective as desired. ∎

Using this, we have

Thm: Let $c_i := c_i(E_{taut}) \in H^{2i}(BU(k); \mathbb{Z})$ (rank k). Then, the classes c_i are algebraically independent for $i = 1, \ldots, k$, & moreover $H^\bullet(BU(k); \mathbb{Z}) \underset{\text{as rings}}{\cong} \mathbb{Z}[c_1, \ldots, c_k]$ ($|c_j| = 2j$).

Pf: Consider the map $f_k : (\mathbb{CP}^\infty)^k \longrightarrow BU(k)$ which classifies $E = \bigoplus_{i=1}^k \pi_i^* L_{taut}$. By prev. Prop, $f_k^* : H^\bullet(BU(k); \mathbb{Z}) \longrightarrow H^\bullet((\mathbb{CP}^\infty)^k; \mathbb{Z}) \underset{\text{Kunneth}}{\cong} \mathbb{Z}[h_1, \ldots, h_k]$ is injective, ($|h_i| = 2$ for each i.)
so need to calculate $\mathrm{im}(f_k^*)$. Now consider the action of the symmetric group Σ_k on $(\mathbb{CP}^\infty)^k$ by permuting factors. The induced action on $H^\bullet((\mathbb{CP}^\infty)^k)$ permutes $(h_1, \ldots, h_n)$. Observe that E is invariant under such an action, that is, $\sigma^* E \cong E$ for any $\sigma \in \Sigma_k$. In particular, $f_k \circ \sigma$ still classifies E, so (by uniqueness of classifying maps up to homotopy) $f_k \circ \sigma \simeq f_k$ i.e., $\sigma^* f_k^* = f_k^*$. Hence the image of f_k^* lands in symmetric polynomials in $h_1, \ldots, h_k$.

Permutes h_i's.

$1 + c_1 + c_2 + \cdots + c_k$

Let's calculate $f_k^*(c(E_{taut})) = c(f_k^*(E_{taut})) = c\left(\bigoplus_{i=1}^k \pi_i^* L_{taut}\right)$

$$\underset{\text{whitney sum}}{=} \prod_{i=1}^k c(\pi_i^* L_{taut}) = \prod_{i=1}^k \pi_i^*(c(L_{taut})) = \prod_{i=1}^k \pi_i^*(1+h)$$

$h_i := \pi_i^* h.$

$$= \underbrace{(1+h_1)\cdots\cdots(1+h_k)}.$$

Hence $f_k^* c_i$ = deg $2i$ part of $\nearrow$ $= \left(\sum_{\substack{J\subseteq\{1,\dots,k\}\\ |J|=i}} \prod_{j\in J} h_j\right) = \sigma_i$ i^{th} elementary symmetric polynomial in $h_1,\dots,h_k$.

Fact: There are no alg. relations between any elementary symmetric polynomials, and any symmetric polynomial in $h_1,\dots,h_k$ can be uniquely written as a polynomial in $\sigma_1,\dots,\sigma_k$

Using this, we learn that $\operatorname{im}(f_k^*) = \{$subring of $\mathbb{Z}[h_1,\dots,h_k]$ gen. by $\sigma_1,\dots,\sigma_k\} \cong$ all symmetric polynomials $\cong \mathbb{Z}[\sigma_1,\dots,\sigma_k]$.

deg $\sigma_i = 2i$.

with $c_i \xmapsto{f_k^*} \sigma_i$.

Hence $H^\bullet(BU(k);\mathbb{Z}) \cong \mathbb{Z}[c_1,\dots,c_k]$. ☒.

Cor: Each char. class $\phi: \mathrm{Vect}^k_{\mathbb{C}}(-) \to H^\bullet(-;\mathbb{Z})$ [of complex rank k bundles] must have the form $E \mapsto q(c_1(E),\dots,c_k(E))$ where q is a polynomial uniquely determined by the class. (q is the element of $\mathbb{Z}[c_1,\dots,c_k] \cong H^\bullet(BU(k);\mathbb{Z})$ given by taking $\phi(E_{taut})$).

Betti #'s : rank $H^i = b_i$.

Cor: $b_{2k+1}(BU(n)) = 0$, & $b_{2k}(BU(n)) = \operatorname{rk} H^{2k}(BU(2k))$

$$= \dim\left(\text{deg. }2k \text{ part of } \mathbb{Z}[c_1,\dots,c_n]\right) \quad |c_i| = 2i.$$

$= \#$ of monomials $c_1^{r_1}\cdots c_n^{r_n}$ of degree $2k = 2(r_1+2r_2+3r_3+\dots+nr_n)$. $r_i \geq 0$

$= \#$ of n-tuples $(r_1,\dots,r_n)$ w/ $k = r_1+2r_2+\dots+nr_n$.

$= \#$ of unordered partitions of k into at most n integers $\{k_1,\dots,k_n\}$ ($k_1\leq k_2\leq\dots\leq k_n$ & $\sum k_i = k$)

via $(r_1,\dots,r_n) \longleftrightarrow \underbrace{r_n}_{k_1} \leq \underbrace{r_n+r_{n-1}}_{k_2} \leq \underbrace{r_n+r_{n-1}+r_{n-2}}_{k_3} \leq \dots \leq \underbrace{r_n+\dots+r_1}_{k_n}$.

The same arguments apply to compute $H^*(BO(k);\mathbb{Z}/2)$ (using $\mathbb{RP}^\infty$ instead of $\mathbb{CP}^\infty$ etc. as usual)

$\Rightarrow$ Thm: $H^*(BO(k);\mathbb{Z}/2) \cong \mathbb{Z}[w_1,\dots,w_k]$ where $w_i := w_i(E_{taut})$, $|w_i| = i$.

(in particular w_i are all alg. independent).

$\Rightarrow$ all char. classes of real vect. bundles of rank k taking values in $H^*(-;\mathbb{Z}/2)$ are polynomials in the Stiefel-Whitney classes.

We won't spell out the details, but a more involved computation shows that, modulo certain 2-torsion elements

$H^*(BO(k);\mathbb{Z}) \cong \mathbb{Z}[p_1,\dots,p_{\lfloor \frac{k}{2} \rfloor}]$ (mod 2 torsion)

certain polynomials in Stiefel-Whitney classes

Pontryagin classes of E_{taut}.

(beginning of another possible paper topic!)

We can also look for char. classes of vector bundles equipped with more structure, e.g., an orientation.

This is what we'll now do. (goal is to define the Euler class of oriented $E \to X$ using a natural class on an oriented bundle called its Thom class).

lives in $H^*(X;\mathbb{Z})$

lives in $H^*(E, E\setminus \underline{0})$.

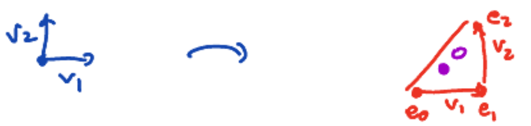
Recall that an orientation of a vector space V dimension n is

$V^\circ := V \setminus 0$

an equivalence class of basis $\mathcal{B} = (v_1,\dots,v_n)$ modulo $\mathcal{B} \sim \mathcal{B}'$ if $\mathcal{B} = T\mathcal{B}'$ w/ $\det(T) > 0$ OR a generator μ_V of $H_n(V, V^\circ; \mathbb{Z})$

$[e_0,\dots,e_n] = \sigma_{\mathcal{B}}$

(Exercise: why is this true? Assign to a basis $\mathcal{B} = (v_1,\dots,v_n)$ a linear n-simplex in V w/ barycenter in 0 w/ $\overrightarrow{e_0e_1} = \vec{v}_1$, $\overrightarrow{e_1e_2} = \vec{v}_2$, ...



$\Rightarrow$ a gen. for $H_n(V, V^\circ;\mathbb{Z})$

check now that if $\mathcal{B} \sim \mathcal{B}'$ then $[\sigma_{\mathcal{B}}] = [\sigma_{\mathcal{B}'}]$.

if $\mathcal{B} \not\sim \mathcal{B}'$ then $[\sigma_{\mathcal{B}}] = -[\sigma_{\mathcal{B}'}]$.)

Similarly the cohomology group $H^n(V, V_0;\mathbb{Z})$ has a preferred generator u_V if V is oriented, by require $\langle u_V, \mu_V \rangle = +1$.

We say a vector bundle $E^n \to X$ is orientable if (rank)

i.e., can find a trivializing cover $\{U_\alpha\}$ s.t. clutching fns. take values in $GL(n)^+$ OR $\mathrm{Frame}(E)$ has a reduction to $GL(n)^+$

E admits a reduction of structure group to $GL(n)^+ \subset GL(n)$

$\Longleftrightarrow$ $\exists$ a section of $\mathrm{Frame}(E) \times_{GL(n)} \left(GL(n)/GL^+(n) \cong \mathbb{Z}/2 \right)$.

$\iff \exists$ a section of the bundle whose fibers are $\{H^n(E_x, (E_x)_0; \mathbb{Z})\}_{x \in X}$, generating each fiber.

construct this bundle? analogous to the bundle $M_{\mathbb{R}} \to M$ we constructed earlier,
or: if $E = \mathrm{Frame}(E) \times_{GL(n)} \mathbb{R}^n$, then consider $\mathrm{Frame}(E) \times_{GL(n)} \underbrace{H^n(\mathbb{R}^n, (\mathbb{R}^n)_0; \mathbb{Z})}_{GL(n) \text{ acts induced by action on } \mathbb{R}^n.}$

$\iff$ A choice $\{u_x \in H^n(E_x, (E_x)_0; \mathbb{Z})\}_{x \in X}$ varying 'continuously';

$E|_U - 0|_U$.

meaning for each x $\exists$ $U \subseteq X$ open containing x & $u_U \in H^n(E|_U, (E|_U)_0; \mathbb{Z})$ restricting along $(E_y, (E_y)_0) \overset{i_y}{\hookrightarrow} (E|_U, (E|_U)_0)$ to u_y for each $y \in U$..