Last time: A rank n vect. bundle is orientable if (among other equivalent characterizations)

∃ a section of the bundle whose fibers are $\{H^n(E_x, (E_x)^\circ; \mathbb{Z})\}_{x \in X}$, generating each fiber.

$V^\circ := V \setminus \underline{0}$

An orientation is a choice of such a section, ⇝ "continuously varying" $\{u_x\}_{x\in X}$ w/ $u_x \in H^n(E_x, (E_x)^\circ)$

$E^\circ := E \setminus \underline{0}$ — image of 0-section

—

Def: A Thom class for an oriented vector bundle $\overset{\text{rank } n}{E} \to X$ is a class $u \in H^n(E, E^\circ)$ (implicitly $\mathbb{Z}$-coeffs.)

with $i_x^* u = u_x$ for each $x \in X$ where $i_x : E_x \hookrightarrow E$ incl. of a fiber

(can also ask for a Thom class w/ $\mathbb{Z}/2$-coeffs, but then don't require E to be orientable; following results all hold w/ $\mathbb{Z}/2$ coeffs. for bundles which are not nec. orientable)

Lem: If such a u exists, then

(a) (Thom isomorphism theorem) The map $\Psi : H^\bullet(X) \xrightarrow{\cong} H^{\bullet+n}(E, E^\circ)$ is an iso.

$\alpha \longmapsto u \smile \pi^*\alpha$ — $\pi^*\alpha \in H^\bullet(E)$, then use rel cup product.

i.e., $H^i(X) \xrightarrow{\cong} H^{i+\text{rank}(E)}(E, E^\circ)$ i.e.,

- $H^k(E, E^\circ) = 0$ for $k < \text{rank}(E)$.
- Any element of $H^n(E, E^\circ)$ has the form $\pi^* f \smile u = f \cdot u$ for f a function on X $f : X \to \mathbb{Z} \leftrightarrow \overset{f}{\overset{\cap}{C^0(X;\mathbb{Z})}}$ which is locally constant $\longleftrightarrow \delta f = 0$.

(b) In particular, by , such a u is unique.. (immediate cor. of (a)).

(any $\tilde{u} \in H^n(E, E^\circ)$ is of the form $\tilde{u} = f \cdot u$, but now

$$\begin{array}{ccc} i_x^* \tilde{u} & = & u_x \\ \| & & \| \\ i_x^*(f \cdot u) & & \\ \| & & \\ f(x)\, i_x^* u & = & f(x)\, u_x \end{array} \qquad \Rightarrow f(x) = 1 \ \forall x.$$

Pf of Lemma:

Observe that one can extend Leray-Hirsch theorem to study fibration pairs over B, i.e., pairs of fibrations $(P, P')$ ($P' \subset P$) whose fibers are $(F, F')$ ($F' \subset F$). Leray-Hirsch in such a setting says:

$$\begin{array}{c} (P, P') \\ \downarrow \\ B \end{array}$$

If $H^*(F,F')$ is free + fin. gen. in each degree and $H^\bullet(P,P') \xrightarrow{res^*} H^\bullet(F,F')$ is surjective,

then choosing classes $\{c_j \in H^{n_j}(P,P')\}$ restrict to a given generating basis $\{\gamma_j \in H^{n_j}(F,F')\}$

"cohomology extension of fibre"

determines an iso. of $H^\bullet(B)$-modules

$$H^*(B)\otimes_R H^\bullet(F,F') \xrightarrow{\cong} H^\bullet(P,P')$$

$$b\otimes\gamma_j \longmapsto \underbrace{\pi^*b\cup c_j}_{\text{"}b\cdot c_j\text{ using module str. of }H^\bullet(B)\text{ on }H^\bullet(P,P')\text{"}}.$$

(Pf is same, or can be deduced from absolute case by studying LES of a pair. — exercise).

Our case: $$\begin{tikzcd} (P,P') \arrow[d] & = & (E,E^0) \arrow[d] \\ B & & X \end{tikzcd}$$ . Note that $H^*(F,F') = H^*(E_x,E_x^0) \cong H^*(\mathbb{R}^n,\mathbb{R}^n\setminus 0)$

$$= \begin{cases} \mathbb{Z} & *=n \\ 0 & \text{else.} \end{cases} \quad \text{free, fin. gen. in each degree.}$$

not Chern class! just $c_j$ w/ $j=1$ as above.

Let $\gamma_1$ be the basis $u_x$ coming from orientation on $E$.

By hypothesis, ∃ 'Thom class' i.e., a class $c_1 = u$ w/ $c_1|_{(F,F')} = \gamma_1$ so restriction map is surjective.

Using this choice of coh. extension of fibre, rel. L-H ⇒

$$\begin{tikzcd}
b\otimes\gamma_1 \arrow[rrr, mapsto, bend left] & H^\bullet(B)\otimes_R \underbrace{H^\bullet(E_x,(E_x)^0)}_{\mathbb{Z}\text{ in degree rank}(E)} \arrow[r, "\cong"] \arrow[d, phantom, "\wr\|"] & H^\bullet(E,E^0) & \pi^*b\cup c_1 = \boxed{\pi^*b\cup u.} \\
b \arrow[u, mapsto] & H^{\bullet+\mathrm{rank}(E)}(B) \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & \\
& b \arrow[rruu, mapsto, bend right] & &
\end{tikzcd}$$

This establishes Thom iso. theorem.

**Existence?**

Thm: If $E \to X$ is orientable, a Thom class always exists. (by above ∃! Thom class for each choice of orientation).

Pf sketch: Inductive argument.

Step 1: A Thom class always exists over $E|_U$, $U\subset X$ if $E|_U$ is trivial.

In that case: $$H^\bullet(E|_U,(E|_U)^0) \cong H^\bullet(U\times\mathbb{R}^n, U\times(\mathbb{R}^n\setminus 0)) \underset{\text{künneth}}{\cong} H^\bullet(U)\otimes H^\bullet(\mathbb{R}^n,\mathbb{R}^n\setminus 0)$$

$$u. \longleftarrow\!\!\shortmid\; 1\otimes \underset{\substack{\uparrow\\ \text{choice of orientation of }\mathbb{R}^n.}}{u_{\mathbb{R}^n}}$$

(exercise: check $u$ is indeed a Thom class for orientation induced by $u_{\mathbb{R}^n}$).

Step 2: Say $E|_{U\cup V}$ orientable, ∃ Thom classes $u_U$ for $E|_U$ and $u_V$ for $E|_V$ compatible w/ chosen orientation of $E$, i.e., $(u_U)_x = (u_V)_x$ when $x\in U\cap V$.

Then ∃ Thom class $u_{U\cup V}$ for $U\cup V$ which restricts to $u_U$ and $u_V$.

by M-V exact sequence for $(E,E^0)$ restr. to $U, V, U\cap V$:

$$\begin{tikzcd}
\overset{0}{H^{n-1}(\cancel{E|}_{U\cap V}, E^0|_{U\cap V})} \arrow[r] & \overset{(\ast)}{H^n(E|_{U\cup V}, E^0|_{U\cup V})} \arrow[r] & \overset{(u_U\ ,\ u_V)}{H^n(E|_U, E^0|_U)\oplus H^n(E|_V, E^0|_V)} \arrow[dll, rounded corners] \\
H^n(E|_{U\cap V}, E^0|_{U\cap V}) \arrow[r] & H^{n+1}(--)\ldots &
\end{tikzcd}$$

0 b/c $\exists$ Thom. class over $U\cap V$ & Thom. iso. applies, $n-1 < \text{rank}(E) = n$

$\mapsto (u_U)|_{U\cap V} - (u_V)|_{U\cap V} = 0$ by hypothesis.

By exactness, $\exists\ u_{U\cup V}$ in $(\ast)$ restricting to $u_U$ and $u_V$ as desired,

Step 3: Inductively as in other proofs use steps 1+2 to deduce existence of Thom classes when $X$ is a finite dim'l CW complex.

(by decomposing $X^k = X^{k-1} \cup \amalg\{e^k_\alpha\}$ & applying to $U = \overset{\circ}{X}{}^k = X^{k-1} \cup \amalg\{\overset{\circ}{e}{}^\alpha_k\}$, $V = \amalg\{\text{int}(e^k_\alpha)\}$, etc.)

$\overset{\circ}{X}{}^k \simeq X^{k-1}$.

Step 4: extend to all CW complexes by 'finite dim'l approx' of any given class.

Step 5: extend to $X$ any space (by 'CW approximation'). ∎.

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Euler class:

Given $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ rank $n$, oriented, real vector bundle, have an inclusion $(X, \phi) \underset{i_X}{\overset{(\underline{0}, \phi)}{\hookrightarrow}} (E, E^0)$.

zero section

Def'n: For $E \to X$ as above with $u \in H^n(E, E^0)$ its Thom class, the Euler class of $E$ is:

$$e(E) := i_X^* u \in H^n(X; \mathbb{Z})$$

$n$ = rank(E).

We can think of $e(E)$ as the image of Thom class under

$$\begin{tikzcd}
X \arrow[r, hook, "\underline{0}", "\text{homotopy equiv.}"'] \arrow[rr, bend right, "i_X"'] & E \arrow[r, "\text{incl.}"] & (E, E^0),
\end{tikzcd}$$

$(E \leftrightarrow (E, \phi))$.

i.e.,
$$\begin{tikzcd}
H^n(E, E^0) \arrow[r, "res^*"] & H^n(E) \arrow[r, "\cong"] & H^n(X), \\
u \arrow[rr, mapsto] & & e(E).
\end{tikzcd}$$

Properties of the Euler class

Lemma: If $E \to X$ has a nowhere vanishing section $s : X \to E$ ($\overset{\text{using a metric on } E}{\Longleftrightarrow}$ $E \cong \underline{\mathbb{R}} \oplus \underset{(\underline{\mathbb{R}})^\perp}{\underset{\uparrow}{E'}}$) then $e(E) = 0$.

"$e(E)$ obstructs existence of a non-vanishing section"

(i.e., $e(F \oplus \underline{\mathbb{R}}) = 0$; note in contrast $w_i/p_i\,(F \oplus \underline{\mathbb{R}}) = w_i/p_i\,(F)$).

Pf: Note: Any two sections $s, s' \in \Gamma(E)$ are homotopic as maps $X \to E$ via homotopy $(1-t)s + ts'$.

In particular, if $E \to X$ has a non-vanishing section $s$, then $s \simeq i_X = (\underline{0}, \phi)$ as maps

$$(X, \phi) \to (E, E^0)$$

$\Rightarrow e(E) = i_X^* u = s^* u$, but since $s$ is nowhere vanishing, $s$ factors as

$$\begin{tikzcd}
(X,\phi) \arrow[rr, "s:=(s{,}\phi)"] \arrow[dr, "s\ (b/c\ s_x \neq 0\ \forall x)."'] & & (E, E^0) \\
& (E^0, E^0) \arrow[ur, "\text{incl.}"'] &
\end{tikzcd}$$

i.e., $s^*$ factors through $H^*(E_0, E_0) = 0$, so $e(E) = 0$. $\boxtimes$.

Say $\underset{\substack{\text{rank } n \\ = n_1 + n_2}}{E} = \underset{\text{rank } n_1}{E_1} \oplus \underset{\text{rank } n_2}{E_2}$ with each $E_i$ oriented $\Longrightarrow$ induces a canonical orientation of $E$.

(fibrewise: if $(e_1, \dots, e_{n_1})$ oriented basis of $(E_1)_x$ & $(f_1, \dots, f_{n_2})$ oriented basis of $(E_2)_x$

$\Rightarrow$ declare $(e_1, \dots, e_{n_1}, f_1, \dots, f_{n_2})$ to be an oriented basis of $E_x$

$\Rightarrow$ a map $or((E_1)_x) \times or((E_2)_x) \to or((E)_x)$).

Using these compatible orientations to define Euler classes:

Prop: $e(E) = e(E_1) \smile e(E_2)$.

Rmk: Similar to, but different in practice from Whitney sum formula for total Chern/Stiefel-Whitney/Pontryagin classes.

note: whereas $w/p\,(\underline{\mathbb{R}}) = 1$ (resp. $c(\underline{\mathbb{C}}) = 1$), $e(\underline{\mathbb{R}}) = 0$, i.e., is not a unit.

So this formula can't always be used e.g., to solve for $e(E_1)$ given $e(E_2)$ & $e(E = E_1 \oplus E_2)$.

---

Pf of proposition:

Let $\pi_i : E \to E_i$ (fiberwise) projection onto $i$th factor, $i = 1, 2$.

gives: $\overline{\pi}_1 : (E, E \setminus E_2) \to (E_1, \underset{\substack{\shortparallel \\ E_1 \setminus \underline{0}}}{E_1^0})$ $\quad \overline{\pi}_2 : (E, E \setminus E_1) \to (E_2, E_2^0)$.

Let $u_i \in H^{n_i}(E_i, E_i^0)$ be the Thom classes of $E_i$ $i = 1, 2$.

<u>Lemma:</u> The Thom class for $E$ (using given orientation), $u$, satisfies:

$$u = \overline{\pi}_1^* u_1 \smile \overline{\pi}_2^* u_2 .$$

$(E\setminus E_1) \cup (E\setminus E_2)$

rel. cup prod. $H^{n_1}(E, E\setminus E_2) \times H^{n_2}(E, E\setminus E_1) \longrightarrow H^{n=n_1+n_2}(E, \overset{\shortparallel}{E^{\circ}})$.

By uniqueness of Thom classes, it suffices to verify both sides are Thom classes & agree on any given fiber $E_x = (E_1)_x \oplus (E_2)_x$.

(Exercise: check that the induced orientation on a direct sum $E_x = (E_1)_x \oplus (E_2)_x$, thought of as an elt. of $H^n(E_x, E_x^{\circ})$, is induced from the ones $(u_i)_x \in H^{n_i}((E_i)_x, (E_i)_x^{\circ})$ $i=1,2$ precisely by $\pi_1^*(u_1)_x \smile \pi_2^*(u_2)_x$.).

<u>Using lemma:</u> $e(E) := i_x^* u$ where

$$\begin{tikzcd} & (E, E\setminus E_1) \arrow[d, "\text{ind.}"] \\ (X,\emptyset) \arrow[ur, "i_x^1"] \arrow[r, "i_x"] \arrow[dr, "i_x^2"'] & (E, E^{\circ}) \\ & (E, E\setminus E_2) \arrow[u, "\text{ind.}"'] \end{tikzcd}$$

$$\Rightarrow i_x^* u \overset{\text{(lem)}}{=} i_x^*(\overline{\pi}_1^* u_1 \smile \overline{\pi}_2^* u_2)$$

$$= (i_x^{1*} \overline{\pi}_1^* u_1) \smile (i_x^{2*} \overline{\pi}_2^* u_2)$$

$$\overset{\text{(exercise)}}{=} e(E_1) \smile e(E_2).$$

---

The Euler class is a (natural) invariant of $(E, \omega)$, though we sometimes leave $\omega$ implicit; & note

bundle, orientation

$$e(E, -\omega) = -e(E, \omega).$$

In particular, since $(-id): \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ is orientation reversing when $n$ is odd.

$$\Rightarrow (E, \omega) \underset{\text{as oriented bundles}}{\overset{(-id)}{\cong}} (E, -\omega) \quad \text{when rank}(E) = \text{odd}.$$

$$\Rightarrow e(E, \omega) = -e(E, \omega) \quad \text{when rank}(E) \text{ is odd}.$$

<u>Cor:</u> If rank$(E)$ is odd, then $2e(E,\omega) = 0$ (ie, $e(E,\omega)$ is 2-torsion).
(will be forced to be zero if no 2-torsion in that coh. group).

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4/9/2021. $E := (E, \omega)$ oriented vec. bundle $E \to X$

$\rightsquigarrow e(E)$ $(e(E,\omega))$ Euler class. $\in H^{\text{rank}(E)}(X; \mathbb{Z})$.

<u>Satisfies:</u>

- if E has a nowhere vanishing section ($\Leftrightarrow$ if $E \cong \underline{\mathbb{R}} \oplus E'$), $e(E) = 0$.
- $e(\underset{\uparrow}{E_1 \oplus E_2}) = e(E_1) \smile e(E_2)$.

  direct sum of oriented vector bundles
- $e(E, -\omega) = -e(E, \omega)$ $\Rightarrow$ If rank(E) odd, $2e(E,\omega) = 0$.

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**Application:** Gysin sequence (a certain exact sequence relating $H^\bullet$(sphere bundle) & $H^\bullet$(base))

Given $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ real vec. bundle, put a metric on it. $\rightsquigarrow$ (unit) disk bundle $\begin{matrix} D(E) \\ \downarrow \\ X \end{matrix} := \{ \overset{x}{v} \in E_x \mid \|v\| \le 1 \}$,

and (unit) sphere bundle $\begin{matrix} S(E) \\ \downarrow \\ X \end{matrix} = \{(x,v) \in E \mid \|v\| = 1\}$ (fibers are $S^{n-1}$'s)

(e.g., $(\overline{B_1(0)}, \partial \overline{B_1(0)}) \xrightarrow{\simeq} (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$)

homotopy equiv.

And there's a map of fibration pairs (over X) $(D(E), S(E)) \overset{\sim}{\hookrightarrow} (E, E^\circ)$

We can write down a LES for cohomology of S(E) as follows:

First, write LES of pair $(D(E), S(E)) \simeq (E, E^\circ)$

$E, E^\circ$ $\qquad E \qquad E^\circ \qquad E, E^\circ$

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(D(E),S(E)) \arrow[r, "j^*"] & H^i(D(E)) \arrow[r] \arrow[d, shift right, "\underline{0}^*"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & H^i(S(E)) \arrow[r] \arrow[d, equal] & H^{i+1}(D(E),S(E)) \arrow[r] & {} \\
{} \arrow[r] & H^{i-n}(X) \arrow[u, "\cong"', "\Phi"] \arrow[r, "\smile e(E)"] \arrow[ur, phantom, "\circlearrowleft"] & H^i(X) \arrow[u, shift right, "\pi^*"', "\cong"] \arrow[r, "\pi^*"'] & H^i(S(E)) \arrow[r] & H^{i-n+1}(X) \arrow[u, "\cong", "\Phi"'] \arrow[r] & \cdots
\end{tikzcd}$$

(Thom. iso) $\qquad \alpha \mapsto \pi^*\alpha \smile u$ $\qquad$ (Thom. iso.

($n = \text{rank}(E)$) $\qquad$ (Claim)

why? By def'n, $\underline{0}^* j^* (\pi^*\alpha \smile u)$
$= \underline{0}^* j^*(\pi^*\alpha) \smile (\underline{0}^* j^* u)$
$= \alpha \smile e(E)$.

Gysin $\Leftrightarrow$ (long exact) Sequence for $\begin{matrix} S(E) \\ \downarrow \\ X \end{matrix}$.

(Rmk: one can establish this type of sequence for any sphere bundle, not just one of the form S(E) for some vector bundle E, via an application of Serre spectral sequence.)

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Rmk: If E is trivial $E \cong \underline{\mathbb{R}}^n$, then $S(E) = X \times S^{n-1}$, $e(E) = 0$, so LES splits & one gets

$$H^i(X \times S^{n-1}) \cong H^i(X) \oplus H^{i-n+1}(X).$$

(which matches Künneth $H^\bullet(X) \otimes H^\bullet(S^{n-1})$)

---

Cor of Euler class: Say $(E, \omega)$ even-dim'l oriented bundle & $2e(E,\omega) \neq 0$. Then, E cannot split as sum of two odd rank oriented bundles.

note: on oriented manifold, $2e(TM)\neq 0 \Rightarrow e(TM)=0$.

If $M$ oriented manifold, we'll call $e(M) := e(TM)$ Euler class of $M$. $\in H^{\dim(M)}(M;\mathbb{Z})$

(using Cor. above)

Exercise: $M$ oriented manifold with $e(M)\neq 0$. Then, $TM$ doesn't admit an odd-dim'l subbundle $\xi \subset TM$. (in particular, $\dim(M)$ is even)

(Hint: case (i): show $\nexists$ orientable odd rank $\xi \subset TM$

(ii) Say $\exists$ $\xi \subset TM$ odd, non-orientable; pull back to a 2-fold cover of $M$ over which $\xi$ orientable to reduce to (i)).

We can also take the characteristic # associated to $e(M)$ (say $M$ cpct, oriented), and:

Thm: $M$ cpct, oriented. $\langle e(M), [M]\rangle = \chi(M)$ ← Euler characteristic of $M$.

There are many ways to define $\chi(M)$.

We won't in class equate $\langle e(M),[M]\rangle$ w/ coh. def'n $\chi(M) = \sum(-1)^i \dim H^i(X)$, (see HW exercise on it).

but here's another way to see Thm, in terms of the definition of $\chi(M)$ involving zeroes of vector fields:

Consider $E \to X$ vector bundle (eventually $E = TM \to M$).

Let $s: X \to E$ any section, $Z \subset X$ zero set of $s$.

Then note

$$\begin{tikzcd}
(X,\emptyset) \arrow[rr, hook, "s\ (\simeq\ i_X = 0)"] \arrow[dr] & & (E,E^\circ) \\
& (X, X\setminus Z) \arrow[ur, "(s,s)"'] &
\end{tikzcd}$$

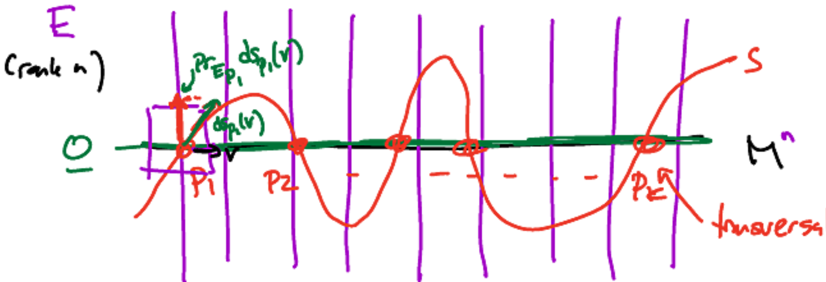
factors:

$\Rightarrow$ (since $s^* = i_X^*$), $e(E)$ is in the image of restriction $H^*(X, X\setminus Z) \longrightarrow H^*(X)$;

i.e., $e(E)$ "admits a representative supported on $Z$."

Say $X = M^n$ cpct oriented, $E \to M$ rank $n$ oriented vector bundle (eventually $E = TM$)

& say $s$ is transverse to zero, as in picture; vanishing at $Z = \{p_1, \ldots, p_k\}$.



transversal intersections; $Z :=$ intersection locus $=$ points $p_1, \ldots, p_k$.

Transversality $\Longleftrightarrow$ at each $p_i$, $pr_{E_{p_i}} \circ ds_{p_i} : T_{p_i}M \xrightarrow{\cong} E_{p_i}$

$$\Big( ds_{p_i} : T_{p_i}M \to \underset{\substack{\cong \\ T_{p_i}M \oplus E_{p_i}}}{T_{(p_i,0)}E} \Big)$$

Assign a sign $\varepsilon(p_i) = \pm 1$ according to whether $pr_{E_{p_i}} \circ ds_{p_i} : T_{p_i}M \xrightarrow{\cong} E_{p_i}$ is orientation preserving or reversing.

Then, we <u>claim</u>: $\langle e(E), [M] \rangle = \sum \varepsilon(p_i) \in \mathbb{Z}$.

(when $E = TM$, $\sum \varepsilon(p_i)$ associated to a transversal (to zero) section $s$ i.e. a vector field, is one way of defining $\chi(M)$! c.f., "index of a vector field.").

To see claim:

$Z = \{p_1, \dots, p_k\}$. Now we know

$$e(E) = j^*(\tilde{e}) \text{ where } \tilde{e} \in H^n(M, M \setminus \overset{\substack{\{p_1,\dots,p_k\} \\ \shortparallel}}{Z}) \underset{\text{excision}}{\cong} \bigoplus H^n(B(p_i), B(p_i) \setminus p_i)$$

$$\underset{\substack{\text{(excise all} \\ \text{but a small} \\ \text{ball around} \\ \text{each } p_i)}}{\cong} \bigoplus_{i=1}^{k} \mathbb{Z} = \mathbb{Z}^k$$

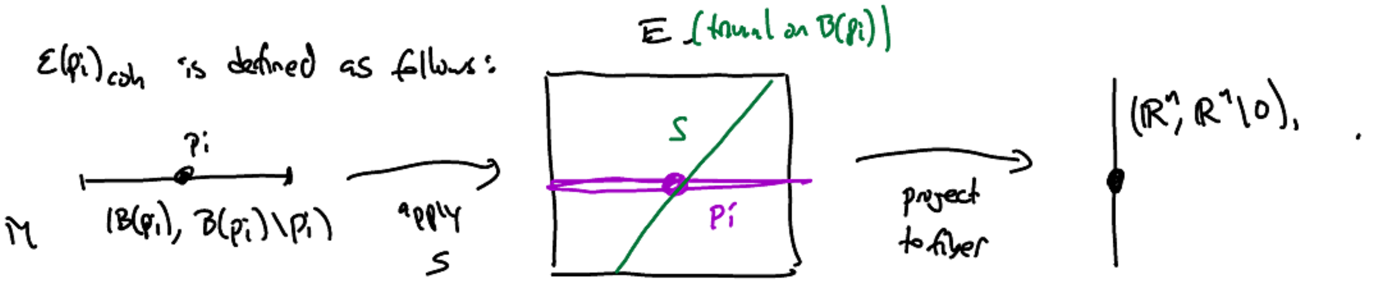
So $\tilde{e} \longleftrightarrow (a_1, \dots, a_k)$, $a_i \in H^n(B(p_i), B(p_i) \setminus p_i) \xrightarrow{\cong} \mathbb{Z}$.
↑ uses orientation on $M$.

call $a_i := \underline{\varepsilon(p_i)_{coh.}}$

$$\begin{tikzcd}
\text{The map } H^n(M, M \setminus Z) \arrow[r] \arrow[d, phantom, "\cong"] & H^n(M) \arrow[d, phantom, "\cong \text{ using orientation (i.e., } \langle -, [M] \rangle)."] \\
\bigoplus_{i=1}^{k} \mathbb{Z} \arrow[r] & \mathbb{Z} \\
(a_1, \dots, a_k) \arrow[r, mapsto] & \sum a_i .
\end{tikzcd}$$

i.e., $\langle e(E), [M] \rangle = \sum \varepsilon(p_i)_{coh.}$. It remains to see $\varepsilon(p_i) = \varepsilon(p_i)_{coh.}$

$\varepsilon(p_i)_{coh}$ is defined as follows:



we pull back a choice of orientation element in $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ (agrees w/ chosen orientation on $E$) to get $\varepsilon(p_i)_{coh}$, and now express as elt. of $\mathbb{Z}$ using orientation on $M$!

<u>Check</u> (exercise, using above description): $\underset{coh.}{\varepsilon(p_i)} = \pm 1$ precisely depending on whether $ds_{p_i}$ is orientation preserving or reversing, i.e., $\varepsilon(p_i)_{coh} = \varepsilon(p_i)$

<u>Cor</u>: If $n$ odd, $E \to M^n$ any oriented rank $n$ bundle on cpct. oriented $\overset{\text{connected}}{\text{mfld}}$ $M^n$, $s$ a section of $E$ transverse to zero, then

$$\underset{\substack{\uparrow \\ \text{signed count.} \\ \text{(using } \pm 1\text{'s as above)}}}{\#\text{zeroes}(s)} = 0$$

(in particular unsigned count is always <u>even</u>).

$$\left(\text{b/c } \; 2e(E)=0 \Rightarrow \underset{\substack{\uparrow \\ H^n(M)=\mathbb{Z}.}}{e(E)}=0 \Rightarrow \langle e(E), [M] \rangle = 0 \right)$$