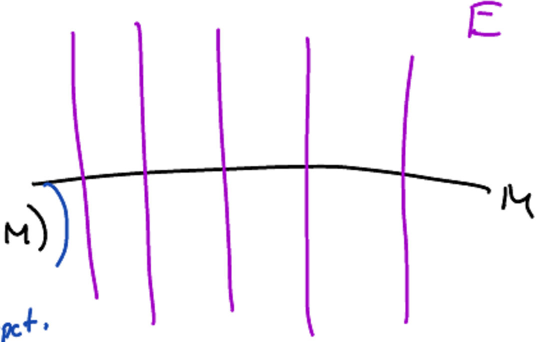
Relation between Thom classes & Poincaré duality

Say M^m oriented cpct (connected for simplicity) manifold, $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$ $\overset{\text{rank } k}{\text{oriented vector bundle}}$, then

E is a non-compact oriented manifold dimension m+k, so we have



non-compact formulation Poincaré duality for E:

$$H^*_c(E) \xrightarrow[\text{P.D.}]{\cong} H_{k+m-*}(E) \left(\xrightarrow{\cong} H_{k+m-*}(M) \right)$$

relate this to other groups we've studied.

b/c M is cpct.

Picking a metric $\langle -,- \rangle$ on E, get an exhaustion of E by cpct. sets $D_R(E) = \{(x,v) \mid \|v\| \le R\}$.

$$\Rightarrow H^*_c(E) \cong \varinjlim_R H^*(E, E \setminus D_R(E))$$

note
$$\begin{tikzcd}
(E, E\setminus D_R(E)) \arrow[r, hook, "\sim", "\text{h.e.}"'] \arrow[d, hook] & (E, E^\circ) \\
(E, E\setminus D_{R'}(E)) \arrow[ur, hook, "\text{h.e.}"'] &
\end{tikzcd}$$
for $R > R'$

(by →).

$$\cong H^*(E, E^\circ)$$

So therefore the Thom class $u \in H^k(E, E_0)$ corresponds to

an element of $H^k_c(E) \cong H_m(E)$ via Poincaré duality for E. what element?

$i_M : M \overset{0}{\underset{\sim}{\hookrightarrow}} E$ homotopy inverse to $\pi : E \to M$

determined by orientation on E as a bundle

dt. by orientation on M

Claim: $\underbrace{\text{P.D. for } E}$ sends Thom class u to $(i_M)_*[M]$ in $H_m(E)$.

determined by orientation on E as a manifold (which in turn is determined by orientation on M and one on E as a bundle).

How to use this fact?

If $K^k \subseteq M^m$ any submanifold, K, M oriented, cpct (assume smooth) ⇒ got a class $\overset{[K]}{\overset{\shortparallel}{i_*[K]}} \in H_k(M)$.

Can ask: how to think about $\text{P.D.}([K]) \in H^{m-k}(M)$ explicitly?

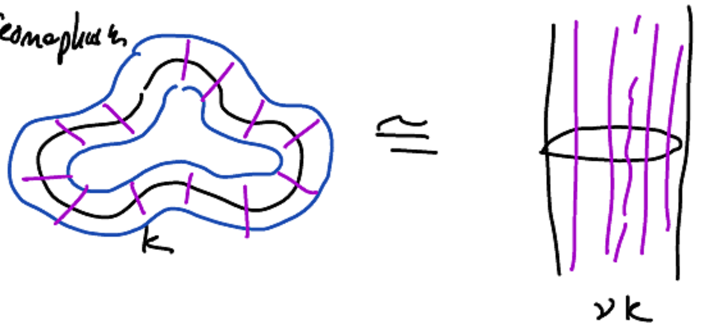
(1) (Tubular neighborhood theorem):

Thm: ∃ open nhood $\mathcal{U}$ of K in M and a diffeomorphism

normal bundle of K in M ($TM|_K / TK$)

$$\begin{tikzcd}
\nu_K \arrow[r, "\cong", "G"'] & \mathcal{U} \\
K \arrow[u, "0"] \arrow[ur, "\text{incl.}"'] &
\end{tikzcd}$$

e.g.,



(2) Given an $\overset{\text{(codim. 0)}}{\text{open}}$ inclusion of m-manifolds $W \overset{i}{\subset} W'$, we prev. asserted one gets

$$\begin{tikzcd}
H_c^*(W) \arrow[r, "i_!"] \arrow[d, "\cong"', "\text{P.D.}"] & H_c^*(W') \arrow[d, "\cong", "\text{P.D.}"'] & \text{"extend by zero" (covariant).} \arrow[l, bend right] \\
H_{m-*}(W) \arrow[r, "i_*"] & H_{m-*}(W')
\end{tikzcd}$$

$\Rightarrow$ in our setting above for $\underset{\text{nhood of } K}{U} \underset{\text{open}}{\overset{i}{\subset}} M$, get:

$$\begin{tikzcd}
u \arrow[dddr, mapsto, bend right, "\text{Thom class of } \nu_K"] & H_c^*(\nu_K) \arrow[r, phantom, "\cong"] \arrow[d, "\cong"', "\text{P.D.}"] & H_c^*(U) \arrow[r, "i_!"] \arrow[d, "\cong"', "\text{P.D.}"] & H_c^*(M) \arrow[r, phantom, "\underset{M \text{ cpct}}{=}"] \arrow[d, "\cong", "\text{P.D.}"'] & H^*(M) & \text{Q: what } \alpha \text{ ?} \arrow[ddll, mapsto, bend left] \\
 & H_{m-*}(\nu_K) \arrow[r, phantom, "\cong"] & H_{m-*}(U) \arrow[r, "i_*"] & H_{m-*}(M) & & \\
 & H_{m-*}(K) \arrow[u, "\cong"] \arrow[ur, "i_*"'] & & i_*[K] & & \\
 & {[K] \in H_k(K) = H_{m-(m-k)}(K)} \arrow[u, phantom, "\cup"] \arrow[urr, mapsto] & & & &
\end{tikzcd}$$

Cor: $i_*[K]$ (sometimes just called $[K]$) $\in H_k(M)$ is P.D. in M to pushforward of Thom class of the normal bundle ν_K to K in M.

Cobordism rings Using what we've developed so far (plus some further indicated topics), want to compute cobordism rings.

Recall: M, M' m-dim'l cpct manifolds. (smooth)

- say M, M' are cobordant if $\exists$ cpct $(m+1)$-manifold-with-∂ W s.t. $\partial W \overset{\text{fixed diffeo}}{\cong} M \amalg M'$.
- If M, M' oriented, say M, M' oriented cobordant if $\exists$ cpct. oriented —— W s.t.
$$\partial W \cong \underset{M \text{ w/ reversed orientation}}{\overline{M}} \amalg M' \text{ as oriented manifolds.}$$

Lemma: The relation of being cobordant resp. oriented cobordant is an equiv. relation. (exercise)

Define: $\Omega_n := \{\text{cpct oriented } n\text{-m'flds}\} / \text{oriented cobordism}$ oriented cobordism grp

$\mathfrak{N}_n := \{\text{cpct } n\text{-m'folds}\} / \text{cobordism}$. (unoriented) cobordism grp.

Group structure? (Focus on Ω_n, $\mathfrak{N}_n$ case is same – and simpler).

note: if $M \underset{\text{oriented diff}}{\cong} M'$ then $[M] = [M']$ in Ω_m, similarly in $\mathfrak{N}_m$ if $M \underset{\text{diff.}}{\cong} M'$.

identity? $0 = [\emptyset]$

addition? $[M] + [M'] := [M \amalg M']$

inverses? note that $M \amalg \overline{M} = \underset{\substack{\uparrow \\ \text{oriented sense}}}{\partial}(M \times [0,1])$ i.e., $[M] + [\overline{M}] = 0$.

$\Rightarrow [\overline{M}]$ additive inverse to $[M]$.

(in $\mathfrak{N}_n$, note that $M \amalg M = \partial(M \times [0,1]) \Rightarrow [M] + [M] = 0$. i.e., $2[M] = 0$

$\Rightarrow [M]$ is its own additive inverse $\Rightarrow \mathfrak{N}_n$ is 2-torsion)

Product structure:

$$M^m, N^n \longmapsto (M \times N)^{m+n},$$ inherits a canonical orientation if M, N oriented.

induces $\Omega_m \times \Omega_n \to \Omega_{m+n}$ $\rightsquigarrow$ $\Omega_* = \bigoplus_{i \geq 0} \Omega_i$ into a graded ring.

(& $\mathfrak{N}_m \times \mathfrak{N}_n \to \mathfrak{N}_{m+n}$.

$\rightsquigarrow \mathfrak{N}_* := \bigoplus \mathfrak{N}_i$ similarly.)

(graded commutative?

$$M^m \times N^n \underset{\text{oriented diff}}{\cong} \overset{\substack{mn}}{(-1)} N^n \times M^m$$)

$\uparrow$ means orientation reverse if mn is odd.

$\Rightarrow [M] \times [N]$

$= (-1)^{mn} [N] \times [M]$.

Basic tools for studying $\Omega_* / \mathfrak{N}_*$: Pontryagin / Stiefel – Whitney numbers:

For any unordered partition $I = \{i_1, \ldots, i_r\}$ of k, have Pontryagin numbers

$$[M^{4k}] \longmapsto P_I [M^{4k}] := \langle \prod_{j=1}^{r} P_{i_j}(M), \underset{\swarrow \mathbb{Z}\text{-fund. class}}{[M]} \rangle \in \mathbb{Z}$$

$\{1,2,1\} = \{1,1,2\}$
$\downarrow$
$P_1 P_1 P_2$
$=$
$P_1^2 P_2$

& Stiefel-Whitney numbers

$$[N^k] \longmapsto w_I [N^k] := \langle \prod_{j=1}^{r} w_{i_j}(N), \overset{\mathbb{Z}/2\text{-fund. class}}{\overset{\swarrow}{[N]}} \rangle \in \mathbb{Z}/2$$

what we've shown is that those associations give group homomorphisms:

$P_I : \Omega_{4k} \to \mathbb{Z}$, $\quad w_I : \mathfrak{N}_k \to \mathbb{Z}/2$. respectively.

HW had an application of w_I to bounding below size of $\mathfrak{N}_k$, for instance.

Similarly, a computation reveals that:

Thm: (e.g., Milnor-Stasheff): The collection $\Big\{\underbrace{\overbrace{\mathbb{CP}^{2j_1}\times\cdots\times\mathbb{CP}^{2j_r}}^{\text{4k-dim'l real manifolds, oriented.}}}_{\boxed{\mathbb{CP}_J}}\Big\}_{J=(j_1,\ldots,j_r)\in p(k)}$ ↑ (unordered) partitions of k

are all linearly independent and non-zero in Ω_{4k}.

$\Rightarrow \operatorname{rank}(\Omega_{4k}) \geqslant \# p(k)$.

A couple ideas of proof:

Basic idea is to show that the matrix $\left(p_I(\mathbb{CP}_J)\right)_{I,J\in p(k)}$ is non-singular.

$$\Rightarrow \prod_{I\in p(k)} p_I : \Omega_{4k} \longrightarrow \mathbb{Z}^{\# p(k)}$$

sends $\{\mathbb{CP}_J\}_{J\in p(k)}$ to a linearly independent collection.

Input: • $p(\mathbb{CP}^{2k}) = (1+h^2)^{2k+1}$ in $\mathbb{Z}[h]/h^{2k+1}$

(so for $i<k$ $p_i(\mathbb{CP}^{2k}) = \binom{2k+1}{i} h^{2i} \Rightarrow \overset{I=\{i_1,\ldots,i_r\}}{p_I}[\mathbb{CP}^{2k}] = \binom{2n+1}{i_1}\cdots\binom{2n+1}{i_r}$

• formula for products. $\boxtimes$

A few oriented cobordism groups (Milnor-Stasheff p. 203): share —.

The key to computing cobordism ring is the following fundamental theorem of Thom:

Def: $\underset{\overset{\shortparallel}{BO(k)}}{G_k(\mathbb{R}^\infty)}$ Grassmannian of k-planes, $\underline{E^k_{taut}} \to G_k(\mathbb{R}^\infty)$.

$\underset{\overset{\shortparallel}{BSO(k)}}{\widetilde{G}_k(\mathbb{R}^\infty)}$ Grassmannian of oriented k-planes, $\underline{\widetilde{E}^k_{taut}} \to \widetilde{G}_k(\mathbb{R}^\infty)$. ↖ oriented rank k vector bundle. (2:1 cover of $BO(k)$ via $\underset{\underset{\mathbb{R}^\infty}{\cap}}{(V,\omega)} \longmapsto V$.)

(classifying space for oriented vector bundles)

Thm: ([Thom]) Fix n. For any $k>n+1$, there is an isomorphism

$$\underset{\text{(n+k)th homotopy group}}{\pi_{n+k}}\Big(\underset{\text{"Thom space"}}{T}(\overset{\text{as above}}{\widetilde{E}^k_{taut}}), t_0\Big) \xrightarrow{\cong} \Omega_n$$

$$\pi_{n+k}\Big(T(\overset{\text{as above}}{E^k_{taut}}), \underset{\text{canonical basepoint in any Thom space.}}{t_0}\Big) \xrightarrow{\cong} \mathfrak{N}_n$$

Goal: explain ingredients of proof, & a way of computing LHS in terms of homology groups.

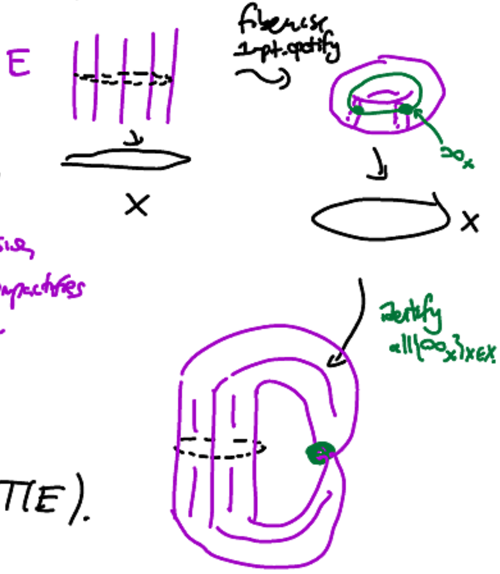
Thom spaces of vector bundles

$E \to X$ real rank k vector bundle. Fixing metric, get $D(E)$ unit disc bundle, $S(E)$ unit sphere bundle, $S(E) \subseteq D(E)$.

Define the Thom space of E to be $T(E) := D(E)/S(E) \cong E/\{(x,v) \mid \|v\| \geq 1\}$.

There's a prefered basepoint $t_0 \in T(E)$ given by $[S(E)] =: t_0$ in $T(E)$.

Can think of forming $T(E)$ in two steps:



- in fiber $\overset{\text{rank } k}{E_x}$, take quotient $\underset{k\text{-disk}}{D(E_x)}/\underset{k-1 \text{ sphere}}{S(E_x)} \cong \underset{\text{sphere of dimension } k \text{ (one point compactifies } E_x\text{).}}{E_x \cup \{\infty_x\}}$

- identify all $\{\infty_x\}_{x \in X}$ to a single point.

We'd like to develop some tools for studying homology (+homotopy groups!) of $T(E)$.

Lemma: E $\overset{\text{rank } k}{\text{oriented}}$, then $\exists$ canonical iso

$$H_{k+i}(T(E), t_0) \cong H_i(B)$$

(homology variant of Thom isomorphism, for Thom spaces).

($\exists$ such an iso. w/ $\mathbb{Z}/2$ coeffs w/o assuming E oriented).

(note as sets $T(E) \cong E \cup t_0$)
'one point cptification of E' (provided X is compact))

Pf sketch:

$$H_{k+i}(T(E),t_0) \cong H_{k+i}(T(E), \overbrace{T(E) - \underline{0}_X}^{T(E)^\circ})$$

(b/c $T(E)^\circ = \amalg E_x^\circ \cup t_0$ is contractible, htpy equiv. to t_0)

$$\underset{\substack{\text{excision}\\ \text{(exercise to)}}}{\cong} H_{k+i}(E, E^\circ) \underset{\substack{\text{Thom iso.}\\ \text{(homology}\\ \text{version)}}}{\cong} H_i(X).$$

given Thom class $u \in H^k(E,E^\circ)$, $- \cap u$:

$$\begin{tikzcd} H_{k+i}(E,E^\circ) \arrow[r] & H_i(E) \arrow[d, "\wr\wr"] \\ & H_i(X) \end{tikzcd}$$

Homotopy groups:

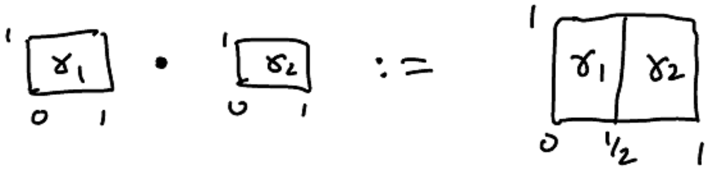
$k>0$, by def'n $\pi_k(\underset{\text{'based space'}}{X, x_0}) := [(S^k, s_0), (X, x_0)]$. in analogy w/ π_1.

have $(S^k, s_0) \simeq (I^k, \partial I^k)$, so can think of:

write $\pi_k(X)$ if x_0 'implicit'

$$\pi_k(X,x_0) := [(\overset{I = [0,1]^k}{I^k}, \partial I^k), (X,x_0)] \quad \& \quad \pi_k(X,A) := [(I^k, \partial I^k), (X,A)].$$

There's a group structure on each π_k, most natural to see via 'concatenate' in first coordinate'



π_k is abelian for $k \geq 2$, via following homotopy:

— 4/14/2021

Today: More about Thom's cobordism theorem:
- homotopy groups + methods of computing (to compute LHS of theorem in some cases) — sketch.
- proof of Thom's thm (sketch)
- applications (sketch)

Homotopy groups are much more difficult to compute (failure of excision). However, in suitable ranges, they can be (partly) computed in terms of homology via (versions of the) Hurewicz theorem.

There's a map

$$\begin{array}{ccc} \pi_r(X) & \xrightarrow{h} & H_r(X;\mathbb{Z}) \\ & & \in \\ [\alpha: S^n \to X] & \longmapsto & \alpha_*[S^n] \end{array}$$

for any (based) $X := (X, x_0)$

can check: h is a group homomorphism.

Can't always be an isomorphism, & for $r=1$ h has to factor through abelianization $\pi_1/[\pi_1,\pi_1]$ b/c H_1 abelian

In Math 540 one proves Thm ($r=1$ Hurewicz thm): h induces an iso. $\pi_1(X)/[\pi_1(X),\pi_1(X)] \xrightarrow{\cong} H_1(X;\mathbb{Z})$.

Now, π_k abelian for $k \geq 2$, & thm states: means $\pi_i(X)=0$ for $i<n$.

Thm: (Hurewicz) If (based) X is $(n-1)$-connected, $n \geq 2$. Then,

$$\tilde{H}_i(X) = 0 \text{ for } i<n \text{ and } h: \pi_n(X) \xrightarrow{\cong} H_n(X).$$

reduced homology ($=H_i$ $i>0$, 0 $i=0$)

Hurewicz homomorphism.

(similarly have Hurewicz for pairs: if (X,A) $n-1$ connected, & A is simply connected non-empty, then ...)

Rmk: There are various ways to show that if X CW complex w/ no i-cells (aside from one 0-cell) for $i<k$, then X is $(k-1)$ connected, i.e., $\pi_i(X)=0$ for $i<k$. (consequence of "cellular approximation of maps,"

c.f., Hatcher) !

$\Rightarrow \pi_i(S^k) = 0$ for $i<k$.

(moment you know a given $f: S^i \to S^k$ up to homotopy misses a point, it factors through $\mathbb{R}^k$, which is contractible)

know: • $\pi_1(S^1) = \mathbb{Z}$, know $\pi_k(S^1) = 0$ for $k>1$ (why? any

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ S^k \arrow[r] \arrow[ur, dashed] & S^1 \end{tikzcd}$$

$k>1$ b/c S^k simply connected, hence is nullhotopic).

The above then tells us: S^k $k \geq 2$, then

$$\pi_i(S^k) = 0 \quad i<k, \quad \text{and} \quad \pi_k(S^k) \cong H_k(S^k) = \mathbb{Z}$$

(in genl $\pi_j(S^k)$ $j>k$ are quite interesting, unlike $H_j(S^k)=0$)

e.g., using cup product on $\mathbb{CP}^2$ we argued that [attaching map in $\mathbb{CP}^2$: $\overset{S^3}{\overset{\shortparallel}{\partial e^4}} \to S^2 = \mathbb{CP}^1$] $\neq 0$ in $\pi_3(S^2)$ (in fact, $\pi_3(S^2) \cong \mathbb{Z}$).

—

However, range of Hurewicz can be extended, provided one forgets some information, [Serre, "Serre classes"].

One instance: let $\mathcal{C}$ denote the class of all finite abelian groups (e.g., $\mathbb{Z}/2$, $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$, NOT $\mathbb{Z}$).

$\mathcal{C}$ is an instance of a <u>Serre class</u> (subclass of all groups closed under subgroups, quotients, extensions)

↳ other examples include {fin. gen. abelian groups}, { p - groups},

A homomorphism $h: A \to B$ is said to be a <u>$\mathcal{C}$-iso.</u> (or iso. mod $\mathcal{C}$), $\mathcal{C}$ any Serre class, if ker A and coker A $\in \mathcal{C}$.

(e.g., for $\mathcal{C}$ = {fin. ab. groups}, $\mathbb{Z}/7 \oplus \mathbb{Z}^5 \xrightarrow[h]{} \mathbb{Z}^5 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$ is a $\mathcal{C}$ iso.)

$$\begin{tikzcd} \mathbb{Z}/7 \arrow[rr, bend left, "0"] & \mathbb{Z}^5 \arrow[r, bend left, "id"] & \mathbb{Z}^5 \end{tikzcd}$$

Similarly, $A=0$ mod $\mathcal{C}$ if $A \in \mathcal{C}$.

Thm: (Mod $\mathcal{C}$ Hurewicz) If X (based) is $(n-1)$-connected mod $\mathcal{C}$ (means $\pi_i(X) = 0$ mod $\mathcal{C}$ $\forall\, i \leq n-1$.), $n \geq 2$. Then,

$\tilde{H}_i(X) = 0$ mod $\mathcal{C}$ for $i<n$ and $h: \pi_n(X) \xrightarrow{\cong} H_n(X)$ mod $\mathcal{C}$.

reduced homology ($=H_i$ $i>0$, 0 $i=0$)

Hurewicz homomorphism.

Have a relative version for (X,A), $A \subset X$, & more genelly a version for maps $A \to X$:

Thm: (generalized Whitehead mod $\mathcal{C}$): $f: A \to X$ map, A, X both simply connected.

(a) $\pi_i(A) \xrightarrow{f_*} \pi_i(X)$ is an iso. mod $\mathcal{C}$ for $i \leq n$ and surjective mod $\mathcal{C}$ for $i=n+1$,

$\Downarrow$

(b) $H_i(A) \xrightarrow{f_*} H_i(X)$ " " " " " " .

Using this, one can show the following claim:

<u>Cor</u>: (Milner-Stasheff): Say X $\overset{\text{finite CW complex (in each dimension)}}{(k-1)\text{-connected}}$, $k \geq 2$. $\mathcal{C} = \{\text{finite abelian groups}\}$

Then $\pi_r(X) \xrightarrow{h} H_r(X;\mathbb{Z})$ is a $\mathcal{C}$ iso. for all $r < 2k-1$

(ordinary iso. for $r \leq k$ by usual Hurewicz)

<u>Pf sketch</u>:

(i) True for spheres S^k. (uses the fact that $\pi_i(S^k) = \begin{cases} \mathbb{Z} & i = k \\ \underline{\text{finite}} & i < 2k-1,\ i \neq k \end{cases}$)

(note $\pi_3(S^2) = \mathbb{Z}$)
$\overset{\shortparallel}{2(2)-1}$.

(ii) If true for X, Y, then true for $X \vee Y$

$\Rightarrow$ true for any $S^{i_1} \vee \cdots \vee S^{i_k}$ $(i_1, \ldots, i_k > 1)$

(iii) General X, ($\underline{\text{fact}}$: $\pi_i(X)$ are finitely generated (omitted)):

Pick generators for all free parts of $\pi_i(X)$, $i < 2k-1$, $\{f_k : S^{n_k} \to X\}$, wedge together:

$$S^{n_1} \vee \cdots \vee S^{n_j} \xrightarrow{\vee f_k} X$$

induces an iso. $\pi_i(S^{n_1} \vee \cdots \vee S^{n_k}) \xrightarrow{\cong} \pi_i(X)$ $\underline{\text{mod } \mathcal{C}}$ for $i < 2k-1$.

By generalized Whitehead mod $\mathcal{C}$, there's also a homology iso. $\underline{\text{mod } \mathcal{C}}$

$$H_i(S^{n_1} \vee \cdots \vee S^{n_k}) \xrightarrow{\cong} H_i(X),$$

reducing us to case (ii). ∎.

— $\mathcal{C} = \{\text{finite abelian groups}\}$ for below:

<u>Cor</u>: (computation of homotopy groups of $T(E \overset{\text{rank } k \text{ oriented}}{\to} X)$)

If $E \to X$ rank k oriented bundle,

$$\pi_{n+k}(T(E), t_0) \xrightarrow{\cong} H_n(X;\mathbb{Z}) \text{ for all } n < k-1, \boxed{\text{mod } \mathcal{C}.}.$$

<u>Pf</u>: $\pi_{n+k}(T(E), t_0) \xrightarrow[\text{}]{\overset{(\text{mod } \mathcal{C})}{\cong}} H_{n+k}(T(E);\mathbb{Z}) \xrightarrow[\substack{\text{Thom} \\ \text{iso} \\ \text{(last class)}}]{\cong} H_n(X;\mathbb{Z})$.

Milnor-Stasheff cor. $(n < k-1)$
so $k+n < 2k-1$.

<u>check</u>: $(k-1)$-connected space.

($\underline{\text{bijection}}$ i cells of $X \longleftrightarrow i+k$ cells of $T(E)$)
for $i > 0$;
cell analogue of Thom $\underline{\text{iso}}$.

<u>Cor</u>: Mod $\mathcal{C}$, $\pi_{n+k}(\widetilde{E}^k_{taut}) \xrightarrow{\cong} H_n(\widetilde{G}_k(\mathbb{R}^\infty);\mathbb{Z})$.

we've computed invariants in class, e.g., we computed $H_*(G_k(\mathbb{R}^\infty), \mathbb{Z}/2)$, $H_*(G_k(\mathbb{C}^\infty), \mathbb{Z})$.

Returning to:

Thm: ([Thom]) Fix n. For any $k > n+1$, there is an isomorphism

$$\pi_{n+k}(T(\tilde{E}^k_{taut}), t_0) \xrightarrow{\cong} \Omega_n$$

$$\pi_{n+k}(T(E^k_{taut}), t_0) \xrightarrow{\cong} \mathfrak{N}_n$$

We'd like to explain some details of the proof. The first is, how to construct a map?

Roughly the idea is to find in $[f: S^n \to T(\tilde{E}^k_{taut})]$ a "smooth" rep; & try to take $f^{-1}(\underline{0}\text{ section})$.

First, $\pi_{n+k}(T(\tilde{E}^k_{taut}), t_0) \cong \pi_{n+k}(T(\tilde{E}^{k,p}_{taut}), t_0)$,

$$\begin{tikzcd} \tilde{E}^{k,p}_{taut} \arrow[d] \\ \tilde{G}_k(\mathbb{R}^{k+p}) \end{tikzcd}$$

for $p >> 0$. (by cellular approx)

smooth vector bundle; smooth manifold

so total space of E is a smooth manifold.

Basic useful def'ns/facts from smooth manifold theory:

M^m, N^n smooth, $f: M \to N$ smooth, recall $y \in N$ <u>regular value of f</u> (or <u>f is transverse to y</u>) if at every $x \in f^{-1}(y)$, $df_x: T_xM \twoheadrightarrow T_yN$. (if $m < n$, this can only happen when $f^{-1}(y) = \emptyset$)

IFT ⇒ at a regular value $f^{-1}(y) \subseteq M$ submanifold dimension $m-n$.

More generally, if $Y \subseteq N^n$ submanifold of <u>codimension k</u> (meaning $\dim(Y) = n-k$). f is <u>transverse to Y</u> if at every $x \in f^{-1}(Y)$,

$$df_x(T_xM) + T_{f(x)}Y = T_{f(x)}N, \text{ or equivalently}$$

$$\begin{tikzcd} T_xM \arrow[r, "df_x"] \arrow[rr, bend right] & T_{f(x)}N \arrow[r, "pr"] & \nu_{f(x)}Y = T_{f(x)}N / T_{f(x)}Y. \end{tikzcd}$$

<u>is surjective</u>:

(Special case: $M \overset{i}{\subset} N \supset Y$, then i is transverse to Y ⟺ $M \pitchfork Y$).

<u>IFT</u> ⇒ If f transverse to Y, then $f^{-1}(Y) \subset M$ submanifold of dimension $m-k$. (codimension k in M)

(if $f = i$ inclusion, $f^{-1}(Y) = M \cap Y$.)

$(T_x f^{-1}(Y) = \ker(T_xM \to \nu_{f(x)}Y)$, so $0 \to T_x f^{-1}(Y) \to T_xM \to \nu_{f(x)}Y \to 0$, ⇒ if M oriented, νY oriented, then $f^{-1}(Y)$ is.

& <u>Lem</u>: orientation of 2/3 of $0 \to A \to B \to C \to 0$ ⇒ orientation of 3rd canonically.

Techniques from smooth topology show that transversality is a "generic condition": by either wiggling f or Y a little, can ensure $\tilde{f}$ is transverse to Y or f is transverse to $\tilde{Y}$.

↖ smoothly homotopic to f, and equals f outside a region.

e.g., [Sard's theorem] ⇒ Regular values of $f: M \to N$ are open dense

⇒ any $y \in N$, $\exists\ \tilde{y}$ arbitrarily nearby regular value.

Now, using these techniques, plus 'smooth approximation' (any continuous $f: Q^q \longrightarrow R^n$ can be approximated up to homotopy by a smooth map (in that region), unchanged outside a nhood of that region).

↖ smooth ↗ in some region

All of these techniques imply:

Thm: (Milnor-Stasheff Thm 18.6): $E \to B$ smooth vec. bdle over a smooth manifold, and let $f: S^m \longrightarrow T(E)$ continuous, sending $s_0 \longmapsto t_0$. Then, f is homotopic to a map $g: S^m \to T(E)$

(recall $E \underset{\text{open}}{\subset} T(E)$ w/ $T(E) - E = t_0$.)

smooth manifold

satisfying:

- g is smooth over $g^{-1}(E) = g^{-1}(T(E) - t_0)$.

  ('smooth approx') open subset of S^m

- g transverse to $B \subseteq E$, submanifold of codimension k.

  ('transversality theory') (zero section inclusion) (note $\nu B = E$)

⇒ $g^{-1}(B)$ is a submanifold of S^m of dimension $m-k$.

Moreover, $g^{-1}(B)$ inherits an orientation from an orientation of E. ↙ ($\Leftrightarrow$ orientation of νB). (S^m is oriented).

- any homotopic $\tilde{g}$ as above induces a cobordant (oriented cobordant if E oriented) manifold:

$$[\tilde{g}^{-1}(B)] = [g^{-1}(B)] \in \Omega_{m-k} \quad (\text{or } \mathfrak{N}_{m-k}).$$

(why? if $g \simeq \tilde{g}$, smoothly approximate the homotopy to get $H: S^m \times [0,1] \longrightarrow T(E)$ which is smooth over $H^{-1}(E)$, & further perturb H to ensure $H \pitchfork B$.

⇒ $H^{-1}(B)$ m'fold w/ boundary, oriented if E is, w/

$$\begin{tikzcd} H^{-1}(B) \arrow[d] \\ {[0,1]} \end{tikzcd} \qquad \partial H^{-1}(B) = \overline{g^{-1}(B)} \amalg \tilde{g}^{-1}(B) \;).$$

over 0   over 1.

smooth rank k bdle over smooth B

— gives a map $\pi_m(T(E), t_0) \longrightarrow \Omega_{m-k}$ (or $\mathfrak{N}_{m-k}$ if E unoriented).

This is Thom's map:

$$\pi_{n+k}(\widetilde{E}^{k,p}_{taut}) \longrightarrow \Omega_n$$

&

$$\pi_{n+k}(E^{k,p}_{taut}) \longrightarrow \mathfrak{N}_n.$$

Why is the map an iso? Let's just show its surjective: Focus on oriented case.

Start with any M^n cpct oriented manifold, $[M^n] \in \Omega_n$.

(1) [Whitney embedding]

$\exists$ smooth embedding $M \hookrightarrow \mathbb{R}^{n+k}$, $k >> 0$. ν_M oriented b/c M, $\mathbb{R}^{n+k}$ are.

(2) [Tubular nhood theorem]: $\exists$ nhood U of M in $\mathbb{R}^{n+k}$ & diffeo.

$$\begin{tikzcd} U \arrow[r, "\cong"] & \nu_M \\ & M \arrow[ul, "i"] \arrow[u, "\underline{0}"'] \end{tikzcd}$$

(3) [classification of ν_M]: ν_M $\cong (TM)^\perp$ inside $T\mathbb{R}^{n+k}|_M \cong \underline{\mathbb{R}^{n+k}}$. , a rank k oriented bundle over M,

so classified by

$$M \xrightarrow{f} \widetilde{G}_k(\mathbb{R}^{n+p}) \text{ any } p \geq k.$$

$$x \longmapsto \{\nu_x M = T_x M^\perp \text{ in } \mathbb{R}^{n+k} \subset \mathbb{R}^{n+p}\}$$

, and have natural maps.

$$\begin{tikzcd} \nu_M \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{E}^{k,p}_{taut} \arrow[d] \\ M \arrow[r, "f"] & \widetilde{G}_k(\mathbb{R}^{n+p}) \end{tikzcd}$$

These maps are smooth, and $\widetilde{f} : \nu_M \to \overline{E}^{k,p}_{taut}$ is transverse to $B = \underline{0} = \widetilde{G}_k(\mathbb{R}^{n+p})$.

(check: this follows from the fact that $\widetilde{f}$ induces an iso.

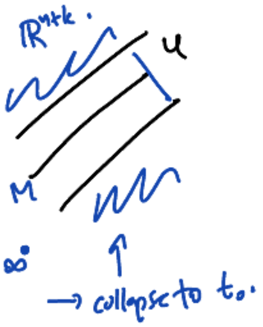
$$\begin{tikzcd} \nu_M \arrow[r, "\cong"] \arrow[d, equal] & f^* \widetilde{E}^{k,p}_{taut} \\ T^{vert}_{(x,0)} \nu_M \arrow[r, "\cong"] & \nu_{(f(x),0)} B \arrow[u, equal] \end{tikzcd}$$

)

(4) The tubular nhood $U \cong \nu M$ induces a map

$$\begin{tikzcd} S^{n+k} = (\mathbb{R}^{n+k} \cup \{\infty\}) \arrow[r] \arrow[dr, "F"'] & T\nu M \arrow[d, "T\widetilde{f}"] \\ & T(\widetilde{E}^k_{taut}) \end{tikzcd}$$

(crushing everything outside U to a point t_0)



can check: F is smooth outside $F^{-1}(t_0)$ (i.e., on U) and $F^{-1}(B = \underline{0}) = M$.

so $[F] \longmapsto [M]$ under Thom's map.

$$\pitchfork$$

$$\pi_{n+k}(T(\widetilde{E}^{k,p}_{taut}), t_0)$$

$\Rightarrow$ $[M] \longmapsto [F]$ gives an (at least one-sided) inverse map $\Omega_n \longrightarrow \pi_{n+k}(\widetilde{E}^{k,p}_{taut})$.
to map in other direction
(called 'Pontryagin-Thom' construction).

4/19/2021. We've so far explained, & sketched part of:

Thm: ([Thom]) Fix n. For any $k > n+1$, there is an isomorphism

$$\pi_{n+k}(T(\widetilde{E}^{k}_{taut}), t_0) \xrightarrow{\cong} \Omega_n \quad \text{(oriented cobordism group)}$$

$$\pi_{n+k}(T(E^{k}_{taut}), t_0) \xrightarrow{\cong} \mathfrak{N}_n \quad \text{(unoriented cobordisms)}$$

Today: Some geometric consequences. First using previously stated results about homotopy groups (of Thom spaces), we can deduce the following from above theorem (focusing on Ω_n)

Cor:

$$\Omega_n \overset{(k,p>>0)}{\cong} \pi_{n+k}(T(\widetilde{E}^{k,p}_{taut}), t_0) \underset{\text{mod } \mathcal{C} = \text{Serre class of finite groups}}{\xrightarrow{\overset{(\text{previously})}{\cong}}} H_n(\widetilde{G}_k(\mathbb{R}^{k+p}); \mathbb{Z}) \text{ for } k,p>>0)$$

$$\begin{tikzcd} \widetilde{E}^{k,p}_{taut} \arrow[d] \\ \widetilde{G}_k(\mathbb{R}^{k+p}) \end{tikzcd}$$

$\uparrow$ $\widetilde{E}^{k,p}_{taut}$ (tautological oriented rank k bundle)

$\widetilde{G}_k(\mathbb{R}^{k+p})$ oriented Grassmannian

$$\Rightarrow \Omega_n \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(\widetilde{G}_k(\mathbb{R}^{k+p}); \mathbb{Q}) \qquad (k,p>>0).$$

Furthermore, we can compute RHS:

Thm: R any integral domain containing $1/2$ (e.g. $\mathbb{Q}$, $\mathbb{Z}[\frac{1}{2}]$, ~). Let $p_i := p_i(E^k_{taut})$, $|p_i| = 4i$. Let $e := e(\widetilde{E}^k_{taut})$, $|e| = k$.

$$H^*(\widetilde{G}_k(\mathbb{R}^\infty); R) \cong \begin{cases} R[p_1, \dots, p_s] & k = 2s+1 \text{ is odd} \\ R[\overset{\text{deg } 4}{p_1}, \dots, p_{s-1}, \overset{\text{deg } k=2s.}{e}] & k = 2s \text{ " even} \end{cases}$$

(recall when k is odd, e is 2-torsion so 0 in R)

$$\cong R[p_1, \dots, p_s, e] / e^2 = p_s.$$

HW: show for any oriented v.b. rank $2s$, E, $e(E)^2 = p_s(E)$.

Assuming Thm:

And $H^i(\tilde{G}_k(\mathbb{R}^{k+p});R) \cong H^i(\tilde{G}_k(\mathbb{R}^\infty);R)$ for $p >> i$.

Cor: $\operatorname{rank}_{\mathbb{Q}} H_n(\tilde{G}_k(\mathbb{R}^{k+p});\mathbb{Q}) = \operatorname{rank}_{\mathbb{Q}} H^n(\tilde{G}_k(\mathbb{R}^{k+p});\mathbb{Q}) \underset{p\text{ large}}{=} \operatorname{rank}_{\mathbb{Q}} H^n(\tilde{G}_k(\mathbb{R}^\infty);\mathbb{Q})$

$\|$

$\operatorname{rank}(\Omega_n \otimes_{\mathbb{Z}} \mathbb{Q})$.

$$\overset{k\text{ large}}{=} \boxed{\begin{cases} 0 & n \neq 4t \text{ some } t \\ \underset{\uparrow}{\#p(t)} & n = 4t \end{cases}}$$

unordered partitions of t. e.g., $\{1,1,2\}$ unordered partition of 4

$\{ \leftrightarrow$

$p_1 p_1 p_2 = p_1^2 p_2$.

We've already exhibited previously a surjection

$$\begin{tikzcd} \Omega_{4t} \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, two heads, "(*)"] & \mathbb{Q}^{\#p(t)} \\ {[\underbrace{\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r}}_{\mathbb{CP}^I}]_{I=\{i_1,\dots,i_r\} \in p(t)}} \arrow[r, mapsto] & \left(p_J(\mathbb{CP}^I)\right)_{J \in p(t)}. \end{tikzcd}$$

$\Rightarrow$ by above, this map $(*)$ is an isomorphism.

Cor: $\Omega_* \otimes \mathbb{Q}$ is a polynomial algebra over $\mathbb{Q}$ w/ generators $\underset{\Omega_4}{\overset{\uparrow}{[\mathbb{CP}^2]}}$, $\underset{\Omega_8}{\overset{\uparrow}{[\mathbb{CP}^4]}}$, $\underset{\Omega_{12}}{\overset{\uparrow}{[\mathbb{CP}^6]}}$, —

Cor: If M^{4k} closed oriented, and all $p_I[M] = 0$ $I \in p(k)$ then $[M] = 0$ in $\Omega_{4k} \otimes_{\mathbb{Z}} \mathbb{Q}$.

$\Rightarrow [M]$ is torsion in Ω_{4k} $\Rightarrow$ for some ℓ, $\underbrace{M \sqcup \cdots \sqcup M}_{\ell \text{ copies}}$ bounds an oriented W.

Shaper varieties on this Theorem:

oriented case:

([M] = 0 in Ω_n) closed, oriented

Thm (Wall): M^n closed oriented, then $M = \partial W^{n+1}$ iff $p_I[M] = 0$ and $w_J[M] = 0$ $\forall I, J$.

$(\Rightarrow \overset{\forall s,}{} \Omega_s \cong \mathbb{Z}^{\oplus a} \oplus \mathbb{Z}/2^{\oplus b}$ for some a, b).

unoriented case: closed unoriented $(\Leftrightarrow [M] = 0$ in $\eta_n)$

Thm (Thom): $M^n = \partial W^n$ iff $w_I[M] = 0$.

Pf. proceeds by exhibiting an injection

constructed using other methods in homotopy theory.

$$\eta_n \cong \pi_{n+k}(TE^{k,p}_{taut}, \infty) \hookrightarrow H_{n+k}(TE^{k,p}_{taut}; \mathbb{Z}/2) \underset{\text{Thom iso.}}{\cong} H_n(G_k(\mathbb{R}^\infty); \mathbb{Z}/2)$$

& check:

$$[M] \longmapsto \langle f^*(-); [M] \rangle : \overset{\overset{w_I}{\uparrow}}{H^n}(G_k(\mathbb{R}^\infty); \mathbb{Z}/2) \to \mathbb{Z}/2$$

where $f: M \to G_k(\mathbb{R}^\infty)$ classifies normal bundle to some embedding. $M \hookrightarrow \mathbb{R}^N$

Computing $H^*(\widetilde{G}_k(\mathbb{R}^\infty); R)$: $BSO(k)$

(b/c oriented rank k vec. bundles $\longleftrightarrow$ $SO(k)$-principal bdles).

Furthermore, we can compute RHS:

Let $e := e(\widetilde{E}^k_{taut})$, $|e| = k$.

<u>Thm</u>: R any integral domain containing $\frac{1}{2}$ (e.g., $\mathbb{Q}$, $\mathbb{Z}[\frac{1}{2}]$, ~). Let $p_i := p_i(\widetilde{E}^k_{taut})$, $|p_i| = 4i$

$$H^*(\widetilde{G}_k(\mathbb{R}^\infty); R) \cong \begin{cases} R[p_1, \ldots, p_s] & k = 2s+1 \text{ is odd} \\ R[p_1, \ldots, p_s, e]/e^2 = p_s & k = 2s \text{ " even} \end{cases}$$

$s = \lfloor \frac{k}{2} \rfloor$

$s = \lfloor \frac{k}{2} \rfloor$ using HW

$= R[p_1, \to p_{\lfloor \frac{k}{2} \rfloor}, e] \Big/ \begin{matrix} e = 0 & k \text{ odd} \\ e^2 = p_{\lfloor \frac{k}{2} \rfloor} & k \text{ even.} \end{matrix}$

We'll compute using a different method than what we used for $H^*(BU(k); \mathbb{Z})$ & $H^*(BO(k); \mathbb{Z}/2)$ (applicable to these prev. computations): <u>Gysin sequence</u>

<u>Pf</u>: Induct on k.

- $k = 1$. $\widetilde{G}_1(\mathbb{R}^\infty)$ is the 2:1 cover of $G_1(\mathbb{R}^\infty) = \mathbb{R}P^\infty$, i.e., $\widetilde{G}_1(\mathbb{R}^\infty) = S^\infty$, which's contractible (check). $\Rightarrow H^*(\widetilde{G}_1(\mathbb{R}^\infty); R) = R$ in degree 0. ✓.

- Assume true for $k-1$.

Have
$$\begin{tikzcd} \widetilde{E}^k_{taut} \arrow[d] \\ \widetilde{G}_k(\mathbb{R}^\infty) \end{tikzcd}$$
tautological (oriented) bundle. The std metric on $\mathbb{R}^\infty$ induces a fiberwise metric on $\widetilde{E}^k_{taut}$ & take unit sphere:

$$\begin{tikzcd} S^{k-1} \arrow[r] & S(\widetilde{E}^k_{taut}) \arrow[d, "\pi"] \\ & \widetilde{G}_k(\mathbb{R}^\infty). \end{tikzcd}$$

<u>Claim</u>: There's a homotopy equivalence

$$\begin{tikzcd} \widetilde{G}_{k-1}(\mathbb{R}^\infty) \arrow[r, shift left, "\overset{f}{\sim}"] \arrow[dr, "incl."'] & S(\widetilde{E}^k_{taut}) \arrow[l, shift left, "\underset{g}{\sim}"] \arrow[d, "\pi"] \\ & \widetilde{G}_k(\mathbb{R}^\infty) \end{tikzcd}$$
, under which $\pi^* \widetilde{E}^{taut}_k \cong \widetilde{E}^{taut}_{k-1} \oplus \underline{\mathbb{R}}$.

incl. induced by $\widetilde{G}_{k-1}(\mathbb{R}^p) \to \widetilde{G}_k(\mathbb{R}^{p+1})$

$V \mapsto V \oplus \mathbb{R}$

in $\mathbb{R}^N \oplus \mathbb{R}$

$\widetilde{G}_k(\mathbb{R}^N)$

$(V, w \in V \text{ unit vec}) \longmapsto w^{\perp}$ in V, a $k-1$ subspace of $\mathbb{R}^N$.

Sketch:

The map g is induced by:

$$\begin{tikzcd}
S(\widetilde{E}_k^{taut}) \arrow[r, "g_N"] \arrow[dr, "\pi"'] & \widetilde{G}_{k-1}(\mathbb{R}^N) \\
& \widetilde{G}_k(\mathbb{R}^N)
\end{tikzcd}$$

The map f is induced by: $V \longmapsto (V \oplus \mathbb{R},\ 0 \oplus 1)$

$\uparrow$ unit vector in $V \oplus \mathbb{R}$.

$$\begin{tikzcd}
\widetilde{G}_{k-1}(\mathbb{R}^N) \arrow[rr, "f_N"'] \arrow[dr] & & S(\widetilde{E}^k_{taut}) \arrow[dl, "\pi"] \\
& \widetilde{G}_k(\mathbb{R}^{N+1}) &
\end{tikzcd}$$

$V \longmapsto V \oplus \mathbb{R}$

exercise: check f, g homotopy inverse, compatible w/ π, incl., etc. (finish claim).

Gysin sequence for $S(\widetilde{E}^k_{taut}) = \overset{BSO(k-1)}{\widetilde{G}_{k-1}(\mathbb{R}^\infty)}$ says: $(BSO(k) = \widetilde{G}_k(\mathbb{R}^\infty))$
(R-coeffs.)

$$\cdots \to \underset{\substack{\uparrow \\ \text{total space}}}{H^{q-1}(BSO(k-1))} \xrightarrow{\delta_*} \underset{\substack{\uparrow \\ \text{base}}}{H^{q-k}(BSO(k))} \underset{\substack{\uparrow \\ e := e(\widetilde{E}^k_{taut})}}{\xrightarrow{\smile e}} H^q(BSO(k)) \xrightarrow{\pi^*} H^q(BSO(k-1)) \xrightarrow{\delta_*} \cdots$$

Obs: since $\pi^* \widetilde{E}_k^{taut} \simeq \widetilde{E}_{k-1}^{taut} \oplus \underline{\mathbb{R}}$, whitney sum formula for p_i says in R (mod 2-torsion),

$$\boxed{\underset{\substack{\shortparallel \\ p_i(\widetilde{E}_k^{taut}) \\ \cap \\ H^*(BSO(k))}}{\pi^* p_i} = \underset{\substack{\shortparallel \\ p_i(\widetilde{E}_{k-1}^{taut}) \\ \cap \\ H^*(BSO(k-1))}}{p_i}}, \quad i \leq \left\lfloor \frac{k-1}{2} \right\rfloor .$$

Case 1: $k = 2s$ even.

$H^*(BSO(2s-1); R)$ inductively equals $R[p_1, \ldots, p_{s-1}]$, so obs $\Rightarrow$ π^* is surjective

$\Rightarrow \delta_* = 0$
(exactness)

$\Rightarrow$ get SES:

$$0 \to H^q(BSO(k)) \xrightarrow{\smile e} H^{q+k}(BSO(k)) \xrightarrow{\pi^*} H^{q+k}(BSO(k-1)) \to 0$$

$\Rightarrow$ $H^{q+k}(BSO(k))$ is gen. by $p_1, \ldots, p_{s-1}$ and e. ✓.

Case 2: $k=2s+1$ odd.

In this case, $e=0$ so Gysin gives a SES:

$$0 \to H^j(BSO(2s+1)) \xrightarrow{\pi^*} H^j(BSO(2s)) \xrightarrow{\delta_*} H^{j-2s}(BSO(2s+1)) \to 0.$$

$\Rightarrow$ π^* injects $H^*(BSO(2s+1))$ into $H^*(BSO(2s))$, sends p_i to p_i.

$\cong$ want $\qquad\qquad\qquad \cong$ inductively

$R[p_1,\dots,p_s] \qquad\qquad R[p_1,\dots,p_s,e]/e^2=p_s$.

Get a map $A^* = R[p_1,\dots,p_s] \longrightarrow H^*(BSO(2s+1)) \xrightarrow{\pi^*} H^*(BSO(2s))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R(p_1,\dots,p_s,e]/e^2=p_s.$

Every element of $H^*(BSO(2s))$ is of the form $a+eb$, $a,b \in R[p_1,\dots,p_s]$.

$\Rightarrow$ $\dim(H^j(BSO(2s))) = \dim A^j + \dim A^{j-2s}$

$\parallel$ SES $\qquad\qquad \parallel$ as a conclusion. $\qquad \Rightarrow$ by rank computation

$\dim(H^j(BSO(2s+1))) + \dim(H^{j-2s}(BSO(2s+1)))$ $\qquad R(p_1,\dots,p_s] \overset{\cong}{\to} H^*(BSO(2s)))$ ∎.

$$\begin{tikzcd}
\Omega_{4t}\otimes_{\mathbb{Z}}\mathbb{Q} \arrow[r, "\cong"] & \mathbb{Q}^{\#p(t)} \\
{[\underbrace{\mathbb{CP}^{2i_1}\times\cdots\times\mathbb{CP}^{2i_r}}_{\mathbb{CP}^I}]_{I=\{i_1,\dots,i_r\}\in p(t)}} \arrow[r, mapsto] & (p_J(\mathbb{CP}^I))_{J\in p(t)}. \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

Observe any homomorphism $\Omega_s \xrightarrow{f} \mathbb{Z}$ induces $\Omega_s \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{f} \mathbb{Q}$ which knows original f, b/c $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is injective.

Cor: Any homomorphism $\Omega_{4k} \xrightarrow{f} \mathbb{Z}$, that is, an association $M^{4k} \longmapsto f(M) \in \mathbb{Z}$ satisfying
closed, oriented

- $f(M \amalg N) = f(M)+f(N)$
- $f(\partial W) = 0$.

↑closed oriented.

can be expressed as a rational linear combination of Pontryagin numbers.

(which linear combination? compute $f(\mathbb{CP}^I)$ for each $I \in p(k)$ & compare $p_J(\mathbb{CP}^I)_{J\in p(k)}$

$\shortparallel$

$$\mathbb{CP}^{2i_1}\times\cdots\times\mathbb{CP}^{2i_r}$$

Ex: M^{4k} cpct., oriented manifold, have a bilinear P.D. pairing, perfect/non-degenerate:

$$H^{2k}(M;\mathbb{Q})\times H^{2k}(M;\mathbb{Q})\xrightarrow{\langle -\cup -,[M]\rangle}\mathbb{Q}$$

since M $4k$-dim'l, this is a symmetric pairing, b/c $\underset{\overset{\uparrow}{H^{2k}}}{\alpha}\cup\underset{\overset{\uparrow}{H^{2k}}}{\beta} = (-1)^{2k\cdot 2k}\beta\cup\alpha=\beta\cup\alpha$.

(In contrast, if $|\alpha|=|\beta|=2k+1$, $\alpha\cup\beta=-\beta\cup\alpha$).

spectral theorem $\Rightarrow$ can diagonalize the symmetric matrix associated to this pairing, no 0 eigenvalues by non-degeneracy; & take its

signature: # positive diagonal entries − # negative diagonal entries.

e.g., $\begin{pmatrix}1&&0\\&1&\\0&&-1\end{pmatrix}$ signature 1 $\quad\begin{pmatrix}1&\\&-1\end{pmatrix}$ signature 0.

$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ signature -2.

Define $\sigma(M^{4k}) := \text{signature}(H^{2k}\times H^{2k}\longrightarrow\mathbb{Q})$.

Prop: $\overset{\text{cpct oriented.}}{M^{4k}}\longmapsto\sigma(M)$ satisfies.

(1) $\sigma(M\amalg N)=\sigma(M)+\sigma(N)$ ✓ Straightfwd.

(2) If $M=\partial W^{4k+1}$, $\sigma(M)=0$, (exercise:
cpct oriented

(3) $\sigma(M\times N)=\sigma(M)\times\sigma(N)$
(uses Künneth)

(1) recall $i: M\hookrightarrow W$, $i_*[M]=0$, b/c $\partial_*[W]=[M]$.

(2) If $\exists$ ½-dim'l $S\subset(V,\langle -,-\rangle)$ sym., non-deg, bilinear
half-dimensional s.t. $\langle s_1,s_2\rangle=0$ (S isotropic) $\Rightarrow$
$\sigma(V,\langle -,-\rangle)=0$.

(3) use LES of $(W,\partial W)$ to note

$$\begin{tikzcd} 0 & H^{n+1}(W,\partial W) \arrow[l] \arrow[d, "\wr\wr\ \text{P.D.}_W"] & H^n(M=\partial W) \arrow[l] & H^n(W) \arrow[l, "i^*"'] & 0 \arrow[l] \\ & H^n(W)^\vee & & & \end{tikzcd}$$

and $i^*(H^n(W))$ is half-dim'l isotropic by (1).

$\Rightarrow$ σ gives (not just a group hom, but an) algebra hom: $\sigma:\Omega_*\longrightarrow\mathbb{Z}$.

$\Rightarrow$ $\sigma|_{\Omega_{4k}}:\Omega_{4k}\to\mathbb{Z}$ is a rational

linear combination of Pontryagin numbers. What linear combination? depends on k, though because of (3) there's an elegant closed form expression in terms of 'multiplicative char.

classes built out of p_i's, called Hirzebruch Signature Theorem.

dimension 4 $(k=1)$: $\Omega_4\otimes\mathbb{Q} \cong \mathbb{Q}\langle[\mathbb{CP}^2]\rangle$.

check: • $\sigma(\mathbb{CP}^2) = 1$. $\Big(H^2(\mathbb{CP}^2) \times H^2(\mathbb{CP}^2) \longrightarrow \mathbb{Q}$

$h^2 \qquad h^2 \longmapsto \langle h^4, [\mathbb{CP}^2]\rangle = 1$.

• $p_1[\mathbb{CP}^2] = \langle p_1(\mathbb{CP}^2), [\mathbb{CP}^2]\rangle$

$= \langle ((1+h^2)^{2+1})_{\deg 4}, [\mathbb{CP}^2]\rangle = \langle 3h^2, [\mathbb{CP}^2]\rangle = 3.$

So $p_1 = 3\sigma$ on $\mathbb{CP}^2$

⇒ Cor: For any 4-m'fld (closed oriented, smooth), M^4, $\sigma(M) = \frac{p_1[M]}{3}$. (special case of signature thm, stated below).

Thm (Hirzebruch's signature theorem): M^{4k} closed, oriented, smooth. Then:

$\sigma(M) = \langle L_k(p_1,\dots,p_k), [M^{4k}]\rangle$, where:

Hirzebruch 'L genus'

Start w/ power series assoc. to $f(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}}$, $f(t) = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \dots + (-1)^{k-1}\frac{2^{2k}\overset{\text{kth Bernoulli \#.}}{B_k}}{(2k)!}t^k + \dots$

⇒ form $f(\underset{\deg 4}{t_1},\dots,\underset{\deg 4.}{t_k}) = f(t_1)f(t_2)\cdots f(t_k)$ power series in $t_1,\dots,t_k$.

Look at homogeneous deg $4k$ part of this power series, symmetric in $t_1,\dots,t_k$

⇒ can be written as a poly. in $\sigma_1,\dots,\sigma_k$.

elementary symmetric poly's in $t_1,\dots,t_k$ $\quad |\sigma_i| = 4i$. (elem. deg i sym. poly.)

⇒ $(f(t_1,\dots,t_k))_{4k} = L_k(\sigma_1,\dots,\sigma_k)$.

This defines $L_k(p_1,\dots,p_k)$.

---

e.g., $\sigma(M^{12}) = \frac{1}{3^3\cdot 5\cdot 7}(62p_3 - 13p_2p_1 + 2p_1^3)[M]$.

⇒ If $H^4(M) = 0$ (so $p_1(M) = 0$) then $\sigma(M)$ must be divisible by 62.

Ex application of such a result: by construction, if one can show

$\exists$ M^{12} top. orientable m'fold w/ $H^4(M)=0$ but $\sigma(M)$ not divisible by 62.

$\Rightarrow$ M^{12} has no smooth structure!

Such examples exist, see [Brieskorn, Hirzebruch]. For instance:

Let $A^{12} = \overline{B_1(0)} \cap (\underbrace{z_1^2+z_2^2+z_3^2+z_4^2+z_5^2+z_6^3+z_7^5=\varepsilon}_{\text{cplx. codim } 1\text{, smooth near } 0}) \subseteq \mathbb{C}^7$; (real 12-m'fold w/ ∂)

& set $M^{12} = A^{12}/\partial A^{12}$.

It turns out this is homeomorphic to a sphere, so $M^{12} \underset{\text{homeo.}}{\cong} A^{12} \cup_{\underset{\text{homeo.}}{\partial A \cong S^{11}}} D^{12}$ is a top. m'fold, check: orientable.

computations: $H^4(M^{12}) = 0$ and $\sigma(M) = -8$, not divisible by 62!

Historical Remark:

[Milnor '56]: First examples of non-diffeomorphic ('exotic') smooth structures on top. manifolds (these examples were 'exotic' S^7's)

[Kervaire '60]: First example of a top. manifold (M^{10}) not admitting a smooth structure.