4/21/2021

Introduction to Spectral Sequences

(some references): • Hutchings, "Notes on spectral sequences"

• Bott & Tu, "Differential forms in algebraic topology"

• Hatcher's "Spectral sequences in algebraic topology"

• Griffiths + Harris, "Principles of Algebraic geometry"

Our plan: • Spectral sequences in homological algebra (today)

• Spectral sequences arising in topology (focus on Leray-Serre spectral sequence of a fibration

$$\begin{array}{c} F \to E \\ \downarrow \\ B \end{array}$$

in cohomology or homology)

describes way in which $H^*(E)$ is built out of, or related to $H^*(B)$ & $H^*(F)$.
(+ some information)

• Applications: computations, Hurewicz (+ Hurewicz mod $\mathcal{C}$), other theoretical results...

Spectral sequence: an alg. gadget for successively 'approximating' a desired homology/cohomology group, starting from some chain-level information (e.g., a filtration) — generalizes LES associated to a SES as we'll see.

LES associated to a SES, revisited (we'll work w/ chain complexes, everything works for cochain cplxes too).

$C_* = (C_*, \partial)$ chain complex.

Say we have a subcomplex $A_* \subset C_*$, means $A_i \subset C_i \ \forall i$, and $\partial(A_i) \subset A_{i-1}$
(i.e., $\partial(A_*) \subset A_*$)

call $A_* = F_0 C_*$..

$\rightsquigarrow$ get a SES of chain complexes $0 \to F_0 C_* \to C_* \to C_*/F_0 C_* \to 0$.

Homological algebra $\Rightarrow$ get a LES

$$\cdots \to H_{i+1}(F_0 C_*) \xrightarrow{\delta} H_i(F_0 C_*) \to H_i(C_*) \to H_i(C_*/F_0 C_*) \xrightarrow{\delta} H_{i-1}(F_0 C_*) \to \cdots$$

δ definition: Given $\alpha \in H_i(C_*/F_0 C_*)$,

• choose a cycle $x \in C_i/F_0 C_i$ w/ $[x] = \alpha$.

• choose a lift $\tilde{x} \in C_i$ of x, so $\partial x = 0$ (in $C_*/F_0 C_*$) means $\partial \tilde{x} \in F_0 C_{i-1}$ (in C_*).

• define $\delta\alpha := [\partial\tilde{x}] \in H_{i-1}(F_0 C_*)$

(δ uses chain-level information).

Suppose we want to compute $H_*(C_*)$ and we know $H_*(F_0 C_*)$ and $H_*(C_*/F_0 C_*)$.

By exactness, above LES splits into:

$$0 \to \operatorname{coker} \delta|_{H_{i+1}(C_*/F_0 C_*)} \to H_i(C_*) \to \ker \delta|_{H_i(C_*/F_0 C_*)} \to 0.$$

i.e., $0 \to \operatorname{coker} \delta \to H_*(C_*) \to \ker \delta \to 0.$ (SES)

Modulo extension problems (i.e., failure of SES to split), this determines $H_*(C_*)$ from $\operatorname{coker}\delta$ & $\ker\delta$. i.e., if SES splits, (e.g., say working over a field $\mathbb{K}$) then $H_0(C_*) = \operatorname{coker}\delta + \ker\delta$.

In other words, the recipe to compute $H_*(C_*)$:

(a) first compute $H_*(F_0 C_*)$ and $H_*(C_*/F_0 C_*)$

(b) consider the 2-term chain complex

$$H_*(C_*/F_0 C_*) \xrightarrow{\delta} H_{*-1}(F_0 C_*),$$

denote its homology by

$$G_0 H_* = \operatorname{coker}\delta \underset{\text{LES}}{\cong} \operatorname{im}(H_i(F_0 C_*) \text{ in } H_i(C_*)).$$

$$G_1 H_* = \ker \delta. \underset{\text{LES.}}{\cong} H_i(C_*)/\operatorname{im}(H_i(F_0 C_*))$$

(note: $F_0 C_* \overset{i}{\hookrightarrow} C_*$ induces $H_*(F_0 C_*) \xrightarrow{[i]} H_*(C_*)$, & can define

$$F_0 H_*(C_*) = [i](H_*(F_0 C_*)) \subset H_*(C_*).$$

(c) Have a SES

$$0 \to G_0 H_* \to H_*(C_*) \to G_1 H_* \to 0,$$

which modulo extensions, determines $H_*(C_*)$.

Filtrations (of modules, chain complexes): R any ring.

A filtered R-module is an R-module A w/ a filtration by submodules, i.e., an increasing sequence of submodules

$$(\cdots \subset F_{p-1}A \subset F_pA \subset F_{p+1}A \subset \cdots) \subseteq A, \quad p \in \mathbb{Z}$$

with $\bigcup F_pA = A$ and $\bigcap F_pA = \{0\}$.

Sometimes indicate by $A, \{F_pA\}_{p\in\mathbb{Z}}$; sometimes just F_pA.

A single submodule $B \subset A$ induces a filtration by:

$$\cdots \subseteq \underset{\substack{F_{-1}A \\ \| \\ \{0\}}}{\{0\}} \subseteq \{0\} \subseteq \{0\} \subset \overset{}{\underset{F_0A}{\overset{\|}{B}}} \subset \underset{F_1A = A}{A} \subseteq A \subseteq A \subseteq \cdots$$

More generally, a filtration is bounded if $F_pA = 0$ for $p \ll 0$ and $F_pA = A$ for $p \gg 0$.

Given a filtered R-module $(A, \{F_pA\}_{p\in\mathbb{Z}}) \rightsquigarrow$ pth associated graded module, defined by

$$G_pA := F_pA / F_{p-1}A.$$

There's a natural SES by def'n:

$$0 \to F_{p-1}A \to F_pA \to G_pA \to 0.$$

In particular, modulo extension issues (e.g., R a field or all SES's as above split),

then G_pA can be used to determine F_pA from $F_{p-1}A$, e.g., in nice cases (such as when $F_\bullet A$ bounded) the G_pA groups inductively determine A.

(Ex of a filtered R-module: $A = C^{\text{analytic}}(\mathbb{R})$ $\mathbb{R}$-module, $F_pA = \{f \in A \mid f^{(i)}(0) = 0 \ \forall i \geq p+1\}$

Note $F_{p-1}A \subseteq F_pA$. Then $G_pA \xrightarrow{\cong} \mathbb{R}$

$[f] \longmapsto f^{(p)}(0)$).

(analogous for cochain cplx)

Def: A filtered chain complex (over R) is a chain complex of R-modules (C_*, ∂) equipped w/ a filtration by submodules $\{F_pC_i\}_{p\in\mathbb{Z}}$ (i.e., $F_pC_i \subset F_{p+1}C_i$) s.t. $\partial(F_pC_i) \subset F_pC_{i-1}$,

i.e., $(F_pC_*, \partial|_{F_pC_*})$ is a subcomplex for each $p \in \mathbb{Z}$.

shorthand for $(\cdots \to F_pC_i \xrightarrow{\partial} F_pC_{i-1} \xrightarrow{\partial} \cdots)$.

In particular, ∂ induces a well-defined differential

$$\partial : G_p C_i \longrightarrow G_p C_{i-1} \qquad \left(G_p C_i := F_p C_i / F_{p-1} C_i \right)$$

get associated graded complex $G_p C_*$. (SES $0 \to F_{p-1} C_i \to F_p C_i \to G_p C_i \to 0$)

The filtration $F_p C_*$ also induces a filtration on homology $H_*(C_*)$ ("induced homological filtration")

via:

$$F_p H_i(C_*) = \{\text{image of } (H_i(F_p C_*) \xrightarrow{\text{need not be injective!}} H_i(C_*)) \text{ inside } H_*(C_*)\}$$

$$= \{\alpha \in H_i(C_*) \mid \exists\ x \in F_p C_* \text{ cycle with } [x] = \alpha \text{ in } H_*(C_*)\}$$

This induces associated graded pieces $\{G_p H_*(C_*)\}_{p \in \mathbb{Z}}$, which in nice cases (e.g., over a field, filtration is bounded) determines $H_i(C_*)$ (by above).

Say our goal is to compute $H_*(C_*)$, & it's easier to compute $H_*(G_p C_*)$ (differential could be much simpler after $/F_{p-1}$!)

Does $H_*(G_p C_*)$ determine $G_p H_*(C_*)$?

(If so, in favorable cases we could compute $H_*(C_*)$ as above).

Case of a subcomplex:

$$\underset{\{0\}}{F_{-1} C_*} \subseteq F_0 C_* \subseteq F_1 C_* = C_*. \text{ we saw that } G_p H_* \text{ is the homology of}$$

$$H_*(G_1 C_*) \xrightarrow{\delta} H_*(G_0 C_*)$$

when there are more terms, we can similarly in nice cases get to $G_p H_*(C_*)$ (& then $H_*(C_*)$) from $H_*(G_p C_*)$ by "successive approximations":

Homology groups associated to a filtered chain complex

$F_p C_*$ a filtered chain complex.

Denote by $E^0_{pq} := G_p C_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}$ associated graded module.

We've shown/mentioned that ∂ on $\{F_p C_*\}$ induces a differential

$$\partial_0 : [\partial] : E^0_{p,q} \longrightarrow E^0_{p,q-1}.$$

Denote by $E^1_{p,q} = H_{p+q}(G_p C_*, \partial_0) = H_q(E^0_{p,*}, \partial_0)$.

"first order approximation to $H_*(C_*)$, or rather to $G_p H_{p+q}(C_*)$."

Next? Now, we can define

$$\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}.$$

by (analogue of way δ was defined above):

any $\alpha \in E^1_{p,q}$ can be represented by a cycle $\tilde{x} \in G_p C_{p+q}$ for $\partial_0 = [\partial]$, lift to a chain $x \in F_p C_{p+q}$ with $\partial x \in F_{p-1} C_{p+q-1}$.

Now, define $\partial_1(\alpha) := [\partial x] \in H_{p+q-1}\left(F_{p-1} C_{p+q-1} / F_{p-2} C_{p+q-1} \right) = H_{p+q-1}(G_{p-1} C_{p+q-1}) = E^1_{p-1,q}$.

check: $\partial^2 = 0 \Rightarrow \partial_1$ is well-defined and $\partial_1^2 = 0$.

Now, consider the homology again:

$$E^2_{p,q} := \frac{\ker(\partial_1 : E^1_{p,q} \to E^1_{p-1,q})}{\operatorname{im}(\partial_1 : E^1_{p+1,q} \to E^1_{p,q})}$$

(when have a 2-step filtration i.e., subcplx $F_0 C_\bullet \subset F_1 C_\bullet = C_\bullet$, $\partial_1 = \delta$ & we saw $E^2_{p,q} = G_p H_{p+q}(C_*)$, but in general this may not be true).

In general, for each non-negative integer r, define the "rth order approximation" to $G_p H_{p+q}(C_*)$ by:

$Z^{r-1}_{p,q}$ "cycles to order $(r-1)$"; i.e., $x \in F_p$ s.t. ∂x vanishes mod F_{p-r}.

$$(\star) \quad E^r_{p,q} = \frac{\{ x \in F_p C_{p+q} \mid \partial x \in F_{p-r} C_{p+q-1} \}}{F_{p-1} C_{p+q} + \partial(F_{p+r-1} C_{p+q+1})}.$$

$F_{p-1} C_{p+q} \cap Z^{r-1}_{p,q} = Z^{r-2}_{p-1,q}$.

"order $(r-1)$ cycles in F_p that happen to live in F_{p-1}" $\longleftrightarrow$ "order $(r-2)$ cycles in F_{p-1}"

$B^{r-1}_{p,q} := \partial(F_{p+r-1} C_{p+q+1}) \cap Z^{r-1}_{p,q}$

"elements of $Z^{r}_{p,q}$ which are ∂(something in F_{p+r-1})"

Comment: $\frac{A}{B}$ above means $\frac{A}{B \cap A}$; shorthand used above (i.e., B need not lie in A)

<u>Lemma:</u> Let (F_pC_*, ∂) be a filtered complex, & define $E^r_{p,q}$ as above. Then,

(a) ∂ on F_pC_* induces a map

$$\partial_r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1} \quad w/ \quad \partial_r^2 = 0.$$

(b) The homology of ∂_r is $E^{r+1}_{p,q}$ as defined above. i.e.,

$$\frac{\ker(\partial_r : E^r_{p,q} \longrightarrow E_{p-r,q+r-1})}{\operatorname{im}(\partial_r : E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q})} \cong E^{r+1}_{p,q}.$$

(<u>warning</u>: ∂^{r+1} on $E^{r+1}_{p,q}$ is <u>not</u> determined by $(E^r_{p,q}, \partial^r)$, need to use chain-level information from (F_pC_*, ∂) to define ∂_r).

(c) $E^1_{p,q}$ (as above) is simply $H_{p+q}(G_p C_*)$.

(d) If filtration F_pC_i is <u>bounded</u> for all i, then $\forall$ p,q,
for any $r >> 0$ (relative to p,q)

$$E^r_{p,q} = G_p H_{p+q}(C_*) \text{ and } \partial_r \equiv 0 \text{ on } E^r_{p,q}.$$

In this case say $E^r_{p,q} =$ "$E^\infty_{p,q}$"

<u>Pf:</u> Idea of defining ∂_r on $E^r_{p,q}$. Given $\bar{x} \in E^r_{p,q}$, lift to

$$x \in Z^{r-1}_{p,q} = \{y \in F_p C_{p+q} \mid \partial y \in F_{p-r} C_{p+q-1}\} \xrightarrow{\partial} F_{p-r} C_{p+q-1}.$$

$$\text{and map} \qquad x \longmapsto \partial x.$$

Check that ∂x induces a well-defined element in $Z^{r-1}_{p-r,q+r-1}/(--) = E^r_{p-r,q+r-1}$; call it $\partial_r \bar{x}$, check well-defined & $\partial_r^2 = 0$.

Rest: (technical) <u>exercise</u>. ∎.

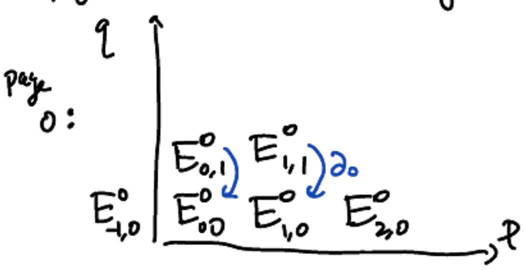
Def'n: A (homological) <u>spectral sequence</u> consists of:

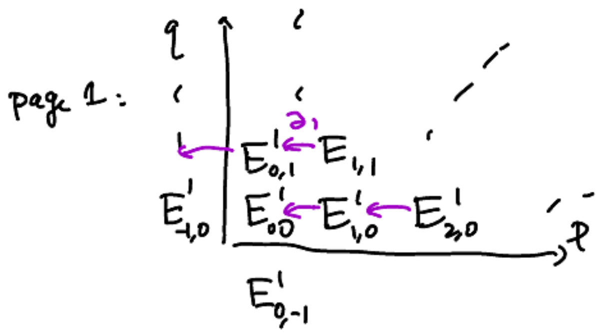
- An R-module $E^r_{p,q}$ defined for each $p,q \in \mathbb{Z}$ and each $r \geq r_0$, $r_0 \in \mathbb{Z}_{\geq 0}$.
- Differentials $\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ s.t. $\partial_r^2 = 0$ and

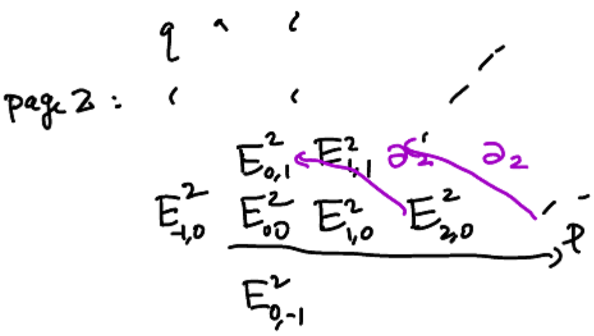
$$E^{r+1}_{p,q} = \frac{\ker(\partial_r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1})}{\operatorname{im}(\partial_r : E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q})}.$$

A spectral sequence <u>converges</u> if, for any p,q for all $r \gg 0$ (rel. p & q) $\partial_r \equiv 0$ on $E^r_{p,q}$ and on $E^r_{p+r,q-r+1}$. $\Rightarrow$ $E^r_{p,q}$ independent of r for $r \gg 0$ (maybe depending on p,q), denote this limiting R-module (if it exists) by $E^\infty_{p,q}$. A spectral sequence <u>collapses</u> or <u>degenerates</u> at <u>page r</u> if on every $r_0 \geq r$, $\partial_{r_0} \equiv 0$ on $E^{r_0}_{p,q}$.
$\Rightarrow E^r_{p,q} = E^\infty_{p,q}$.

We call $\{E^r_{p,q}, \partial_r\}$ the "<u>rth page</u> of spectral sequence, & we can draw a given page at a time in a grid:







What we've shown today is:

Prop: Let $(F_p C_*, \partial)$ be a filtered complex. Then $\exists$ a spectral sequence (S.S.) $(E^r_{p,q}, \partial_r)$ defined for $r \geq 0$ with $E^1_{p,q} = H_{p+q}(G_p C_*)$.

If filtration is bounded, the S.S. converges to $E^{\infty}_{p,q} = G_p H_{p+q}(C_*)$.

— $\boxed{\text{Bonus material}}$: constructing the S.S. of a filtration via exact couples (c.f. [Hatcher S-S], [Bott-Tu]).

Another way to think about how to construct such a spectral sequence is via exact couples [Massey].

An exact couple is a pair A, B of R-modules along with a diagram

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

which is exact at each entry. (i.e., $\operatorname{im} i = \ker j$, etc.)

$\Rightarrow$ $d := jk : B \to B$ satisfies $d^2 = jkjk = 0$.

Given an exact couple, we can define a new exact couple, called the derived exact couple:

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd}$$

Via: $B' = H(B, d = jk)$. $A' = i(A) \subset A$.

- $i'(\underbrace{ia}_{\in A'}) := i(ia)$

- Given $a' \in A'$, pick a w/ $ia = a'$. Now $ja \in B$ satisfies $d(ja) = jkja = 0$, hence is a cycle for d. Define $\boxed{j'(a') := [ja].}$ (Well-defined? (if $\bar{a}$ w/ $i\bar{a} = a'$, then $i(a - \bar{a}) = 0$, so $(a - \bar{a}) \underset{\text{(exactness)}}{=} k(s)$. Hence $j(a - \bar{a}) = jks = ds$ i.e., $ja = j\bar{a} + ds$ i.e., $[ja] = [j\bar{a}]$).

- Given $b' \in B'$, pick $b \in B$ with $db = 0$, $[b] = b'$. i.e., $jkb = 0$.
  Hence $kb = is$ (exactness), i.e., $kb \in i(A) = A'$, so define $k'(b') := kb$.
  (well-defined? if have another $\bar{b}$ with $[\bar{b}] = b'$, then
  $\bar{b} = b + d\tau = b + jk\tau \Rightarrow k\bar{b} = kb + \underbrace{\cancel{kjk\tau}}_{0} \Rightarrow k\bar{b} = kb$.

Lemma: If $\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$ exact couple, then the derived couple $\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd}$ is also exact.

(Pf: homological algebra argument, exercise or see [Hatcher SS.] [Bott-Tu]).

In particular can iterate to get exact couples $\begin{tikzcd} A^r \arrow[rr, "i^r"] & & A^r \arrow[dl, "j^r"] \\ & B^r \arrow[ul, "k^r"] & \end{tikzcd}$ with $B^r = H^{\cdot}(B^{r-1}, d^{r-1} = j^{r-1}k^{r-1})$

Given a filtered chain complex $(F_p C_*, \partial)$, consider

$$A^0 = \bigoplus_p F_p C_* \,, \qquad B^0 = \bigoplus_{p \in \mathbb{Z}} G_p C_* \,,$$

and $i_0 : A^0 \longrightarrow A^0$ induced by $\underset{\substack{\cap \\ \text{domain } A_0}}{F_{p-1} C_*} \xhookrightarrow{\text{ind.}} \underset{\substack{\cap \\ \text{range } A_0}}{F_p C_*}$.

$\exists$ a SES of chain complexes

$$0 \to (A^0)_* \xrightarrow{i_0} (A^0)_* \xrightarrow{j_0} (B^0)_* \longrightarrow 0 \,;$$

inducing a LES:

$$\cdots \to H_*(A^0) \xrightarrow{i_* = i_1} H_*(A^0) \xrightarrow{(j_0)_* \,=\, j_1} H_*(B^0) \xrightarrow{\delta_* \,=\, k_1} H_{*-1}(A^0) \to H_{*-1}(A^0), \text{ i.e.,}$$

an exact couple

$$\begin{tikzcd} A^1 \arrow[rr, "i_1"] & & A^1 \arrow[dl, "j_1"] \\ & B^1 \arrow[ul, "k_1"] & \end{tikzcd}$$

$A^1 = \bigoplus_p H_*(F_p C_*)$. with $i_1 = (i_0)_*$, $j_1 = (j_0)_*$, $k_1 = \delta_*$.

induced by $\mathrm{incl}_* : H_*(F_{p-1}C_*) \to H_*(F_p C_*)$.

$B^1 = \bigoplus_p H_*(G_p C_*)$.

induced by $\delta : H_*(G_p C_*) \longrightarrow H_*(F_{p-1} C_*)$.

Now, we can iteratively derive to get

$$\begin{tikzcd} A^r \arrow[rr, "i_r"] & & A^r \arrow[dl, "j_r"] \\ & B^r \arrow[ul, "k_r"] & \end{tikzcd} \,;$$

& by convention $(B^r, d_r) =: (E^r, \partial_r)$

All of these split into graded pieces $A^r_{p,q}$ where $A^1_{p,q} = H_{p+q}(F_p C_*)$,

$B^1_{p,q} = H_{p+q}(G_p C_*)$; note $k_1 : B^1_{p,q} \to A^1_{p-1,q}$, $i_1(A^1_{p,q}) \subset A^1_{p+1,q-1}$, j_1 preserves (p,q).

$\Rightarrow$ inductively, $A^{r+1}_{p,q} = i_r(A^r_{p,q})$, $\boxed{E^{r+1}_{p,q} := B^{r+1}_{p,q}} = \dfrac{\ker(\overset{\partial_r}{\overset{\shortparallel}{d_r}} : B^r_{p,q} \longrightarrow B^r_{p-r,q+r-1})}{\operatorname{im}(d_r : B^r_{p+r,q-r+1} \longrightarrow B^r_{p,q})}$, $d_{r+1} = j_{r+1} k_{r+1}$

Inductively, if $k_r : B^r_{p,q} \longrightarrow B^r_{p-r,q+r-1}$, $i_r : A^r_{p,q} \to A^r_{p+1,q-1}$, $j_r : A^r_{p,q} \to A^r_{p,q}$.

$$k_{r+1}(s) := \left\{ \begin{array}{l} \text{pick } x \in \ker(d_r) \subset B^r_{p,q} \text{ w/ } [x] = s \text{, and take } k_r(s) \subset A^r_{p-r,q+r-1}. \text{ This is} \\ \text{in the image of } i_r \text{, so lands in } i_r(A^r_{p-r-1,q+r}) = A^{r+1}_{p-(r+1),q+(r+1)-1} \end{array} \right\}$$

This ensures $d_r = \partial_r$ has bidegree $(-(r+1), r)$ as desired.

4/23/2021

Last time:

Prop: Let $(F_p C_*, \partial)$ be a filtered complex. Then $\exists$ a spectral sequence (S.S.) $(E^r_{p,q}, \partial_r)$ defined for $r \geq 0$ with $E^1_{p,q} = H_{p+q}(G_p C_*)$.

If filtration is bounded, the S.S. converges to $E^\infty_{p,q} = G_p H_{p+q}(C_*)$.

determines $H_*(C_*)$ in nice cases (e.g., over a field or if no extension problems, e.g., if every $G_p H_{p+q}$ is proj., every SES splits)

Example: using spectral sequences, can prove that cellular homology = singular homology.

X CW cplx. Define a filtration $F_p C_*(X) := C_*(X^p)$ (note: is submodule of $C_*(X)$)

$(X^0 \subset X^1 \subset X^2 \subset \cdots) \subseteq X$. p-skeleton

$\rightsquigarrow$ associated graded (on chain level):

$$E^0_{p,q} = G_p C_{p+q}(X) = C_{p+q}(X^p) / C_{p+q}(X^{p-1}) = C_{p+q}(X^p, X^{p-1}), \text{ with}$$

$\partial_0 :=$ usual ∂ on relative chains.

Take homology of ∂_0:

$$E^1_{p,q} := H_{p+q}(X^p, X^{p-1}) = \begin{cases} C^{cell}_p(X) & q = 0 \ = \bigoplus_{\alpha \in p\text{-cells in } X} \mathbb{Z}\langle e^p_\alpha \rangle \\ 0 & q \neq 0 \end{cases}$$

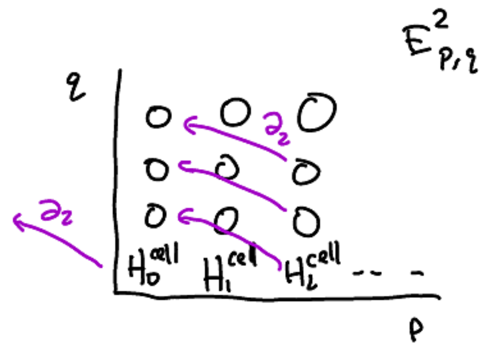
The cellular differential

$$\partial_{cw} : C^{cell}_p(X) \to C^{cell}_{p-1}(X) \text{ is } \overset{\text{(by def'n)}}{\text{the map}}\ H_p(X^p, X^{p-1}) \to H_{p-1}(X^{p-1}, X^{p-2})$$

is the map induced by the LES of the triple (X^p, X^{p-1}, X^{p-2}) (& can compute it using e.g., degrees of attaching maps).

Check from definitions: $\partial_{cw} = \partial_1$ on $E^1_{p,q}$. (exercise.

$$\Rightarrow E^2_{p,q} = \begin{cases} H^{cell}_p(X) & q = 0 \\ 0 & q \neq 0. \end{cases}$$



$E^2_{p,q}$ is therefore supported in a single row (and hence so is $E^r_{p,q}$, which is a quotient of a subgroup of $E^{r-1}_{p,q}$ which is ---- $E^2_{p,q}$)

whereas ∂_r for $r \geq 2$ goes up in row number

$\Rightarrow$ for $r \geq 2$ the domain or codomain of ∂_r is always 0..

$\Rightarrow \partial_r \equiv 0$ for $r \geq 2$ & s.s. collapses / degenerates at page 2.

$\Rightarrow \quad E^\infty_{p,q} = \begin{cases} H_p^{cell}(X) & q \neq 0 \\ 0 & q \neq 0. \end{cases}$ If X is finite-dim'l so filtration bounded, then

$\Rightarrow$ (up to extension issues which can be solved in this case) $\quad H_p(X) = \bigoplus_{i+j=p} E^\infty_{i,j} = H_p^{cell}(X)$

(Then, additional arguments, i.e., taking direct limits $\Rightarrow H_p^{cell}(X) = H_p(X) \quad \forall$ CW cplxes X).

Example: A bi-(chain) complex is a collection of R-modules $\{C_{p,q}\}_{p,q \in \mathbb{Z}}$.

w/ $d_1 : C_{p,q} \to C_{p-1,q}$, $\quad d_2 : C_{p,q} \to C_{p,q-1}$ each satisfying $(d_1)^2 = (d_2)^2 = 0$,

and further: $d_1 d_2 + d_2 d_1 = 0$

(sub-ex: C_*, D_* chain complexes $\Rightarrow$ a bicomplex $\{C_p \otimes D_q\}_{p,q}$ w/ $d_1 = \partial_{C_*} \otimes id_{D_*}$,

$d_2(\alpha \otimes \beta) = (-1)^{deg(\alpha)} \alpha \otimes \partial_{D_*} \beta \quad$ i.e., $d_2 = (-1)^{deg(-)} id \otimes \partial_{D_*}$.).

There's an associated total chain complex $\quad C_*$:

$C_k := \bigoplus_{i+j=k} C_{i,j}$ with differential $\partial = d_1 + d_2$. (note $\partial^2 = d_1^2 + d_1 d_2 + d_2 d_1 + d_2^2 = 0$).

(in sub-ex:, this gives the "tensor product chain complex" of C_* & D_*,

i.e., $C_* \otimes D_*$, $\partial_{C_*} \otimes id + (-1)^{deg(-)} id \otimes \partial_{D_*} = \partial_{C_* \otimes D_*}$).

The fact that C_* came from a bicomplex can be used to define

a filtration:

$F_p C_k := \bigoplus_{\substack{i+j=k \\ i \leq p}} C_{i,j}$. Note $(G_p C_k, \partial|_{G_p C_k}) = (C_{p,k-p}, d_2)$.

($\partial = d_1 + d_2$ preserves $F_p C_*$)

$\Rightarrow$ get a spectral sequence converging (under boundedness hypotheses) to $(G_p H_{p+q}(C_*))$ w/ $E^1_{p,q} = H_q(C_{p,*}, d_2)$,

w.r.t. $\partial = d_1 + d_2$

and $\partial_1 = [d_1]$.

Rmk: there's another filtration, filtering by the other bidegree; giving another spectral sequence! often useful to use both spectral sequences.

$\boxed{\text{Case of } C_* \otimes D_*}$: $d_2 = (-1)^{deg(-)} id \otimes \partial_{D_*}$, so $E^1_{p,q} = C_p \otimes H_q(D_*)$. (UCT for homology, say C_* free or over field)

with $\partial_1 = \partial_{C_*} \otimes id_{H_*(D_*)}$, so can further compute that

$$E^2_{p,q} = H_p(C_* \otimes H_q(D_*)) \underset{\substack{\text{UCT} \\ \text{homology} \\ \text{(over field)}}}{=\!=\!=} H_p(C_*) \otimes H_q(D_*)$$

Now an elt. of $E^2_{p,q}$ can be represented by a sum of elements of the form $\alpha \otimes \beta$ where α cycle in C_p, β cycle in D_q. $\Rightarrow$ $\alpha\otimes\beta$ gives a cycle for

$$\partial_{tot} = \partial_{C_*} \otimes id_{D_*} + (-1)^{\deg(-)} id_{C_*} \otimes \partial_{D_*}$$

$\Rightarrow$ all ∂_r (induced by ∂_{tot}) on such elements vanish, so S.S. collapses at E^2.

$\Rightarrow E^\infty_{p,q} = E^2_{p,q}$, & the obvious map $\bigoplus_{p+q=k} H_p(C_*) \otimes H_q(D_*) \to H_k(C_* \otimes D_*)$

is an isomorphism over a field. ($\boxed{\text{Algebraic Künneth theorem}}$).

The Leray–Serre spectral sequence of a fibration

As an application of the above algebraic machinery for extracting spectral sequences from filtrations, we'll sketch:

generalization of fiber bundle $\begin{smallmatrix} E \\ \pi\downarrow \\ B \end{smallmatrix}$ where one just requires HLP to hold (& also only for maps from disks).

Thm: ((Leray-)Serre Spectral sequence) $\boxed{\text{denote fibers of } \pi: E \to B \text{ by } F_x := \pi^{-1}(x).}$

$\pi: E \to B$ any Serre fibration. Then, $\exists$ a spectral sequence $\{E^r_{p,q}, \partial_r\}$ defined for $r \geq 2$, with

$$E^2_{p,q} = H_p(B; \{H_q(F_x)\}_{x\in B})$$

'homology w/ coefficients in the 'local coefficient system' (bundle) of homologies of fibers.'.

(focus on special case: if $\pi_1(B) = 0$ or 'local coeff. system is trivial' $\longleftrightarrow$ '$\pi_1(B)$ acts trivially on $H_q(F)$').
B path connected.

then, $E^2_{p,q} = H_p(B; H_q(F)) \underset{\text{UCT homology (if over a field } k)}{\cong} H_p(B) \otimes_k H_q(F)$),

fiber over basepoint

converging to

$$E^\infty_{p,q} = G_p H_{p+q}(E) \quad (\text{for some filtration } F_p \text{ on } H_*(E)).$$

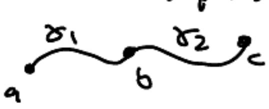
Interlude on local coefficient systems:

X top. space, $\Pi X :=$ fundamental groupoid of X (category)

$\text{ob}\,\Pi X = \{x \in X\}$

$\text{hom}(x,y) = \{\text{homotopy classes } [\gamma] \text{ rel. end points} \mid \gamma: I \to X, \ \gamma(0) = x, \ \gamma(1) = y\}$. composition: concatenation of paths.

i.e., $\text{hom}(x,x) = \pi_1(X,x)$ w/ its group str. induced by composition.



$[\gamma_1]\circ[\gamma_2] = [\gamma_1\cdot\gamma_2]$

A <u>local coefficient system</u> (or local system of abelian gps). is a functor $F : \Pi X \to Ab$ ← cat. of abelian groups

e.g.,

$x \longmapsto G_x := F(x)$. 'fiber over x of F'

$[\gamma]: x \to y \longmapsto G_x \xrightarrow{F([\gamma])} G_y$ 'parallel transport homomorphism' (compat. w/ composition).

$[\text{const}_x] \longmapsto G_x \xrightarrow{id} G_x$

F is trivial if it's constant; (on objects & morphisms) meaning $F(x) =$ some fixed G $\forall x$

$F([\gamma]) = id: G \to G \quad \forall [\gamma]$.

<u>Lemma:</u> X path connected, $* \in X$ basepoint, then

$$\{\text{local coeff. systems}\} \xrightarrow{\cong} \{\text{modules over } \mathbb{Z}[\pi_1(X,*)]\}.$$

(or abelian gps. w/ an action of $\pi_1(X,*)$)

$F \longmapsto F(*)$, w/ action
$\overset{\shortparallel}{G}$ $\quad hom(*,*) = \pi_1(X,*) \xrightarrow{F} hom(G,G)$.

This is an equivalence b/c when X is path-connected, the sub cat. $\{*\}$ is equivalent to ΠX. (i.e., any $p \in X$ is <u>isomorphic</u> in ΠX to $*$).

<u>exercise</u>.

<u>Def:</u> Given a local coeff. system $\Pi X \xrightarrow{F} Ab$, written shorthand as $\mathcal{G} = \{\overset{F(x)}{\overset{\shortparallel}{G_x}}\}_{x\in X}$

↑ abuse of notation, need to remember $F([\gamma])$ $\forall [\gamma]$!

can define $H_*(X;\mathcal{G})$ <u>homology w/ local coefficients</u>.

$$C_p(X;\mathcal{G}) := \bigoplus_{\sigma:\Delta^p\to X} G_{\sigma([1,\dots,0])}\langle\sigma\rangle;$$

↖ element here is $g\langle\sigma\rangle$.

can define differential by observing that if $\sigma : \Delta^p \to X$, then

$$\underset{i\text{th face}}{\partial_i}\sigma(\underbrace{[1,\dots,0]}_{\vec{e}_0}) = \begin{cases} \sigma(\overbrace{[0,1,0,\dots,0]}^{\vec{e}_1}) & i = 0. \\ \sigma(\underbrace{[1,\dots,0]}_{\vec{e}_0}) & i \neq 0 \end{cases}$$

If γ denotes the straight-line path in Δ^p from $\vec{e}_0$ to $\vec{e}_1$, can define

$$\partial(g\langle\sigma\rangle) = \underbrace{F([\sigma\circ\gamma])}(g)\langle\partial_0\sigma\rangle + \sum_{i>0}(-1)^i g\langle\partial_i\sigma\rangle$$

$$\text{maps } G_{\sigma(\vec{e}_0)} \to G_{\sigma(\vec{e}_1)} \qquad G_{\sigma(\vec{e}_0)} \qquad\qquad G_{\partial_i\sigma[\vec{e}_0]} = G_{\sigma[\vec{e}_0]}$$

$$G_{\sigma(\vec{e}_1)} = G_{\partial_0\sigma(\vec{e}_0)}.$$

check $\partial^2 = 0$, call homology $H_p(X;\mathcal{G})$.

Lemma: (1) If $\mathcal{G}$ is trivial, i.e., $\mathcal{G} = \{G\}_{x\in X}$ w/ $F([\gamma]) = id_G$,, then $H_*(X;\mathcal{G}) = H_*(X;G)$.

(2) If X simply connected, all local coeff. systems are trivial.

(3) $\mathcal{G} = \{G_x\}$ local coeff. system $\longleftrightarrow$ M corresp. $\mathbb{Z}[\pi_1(X,*)]$ module (X path connected)

$$\Rightarrow H_*(X;\mathcal{G}) \cong H_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} M.$$

univ. cover

module over $\mathbb{Z}[\pi_1(X)]$ by deck transformations

$H_*(X, M = \mathbb{Z}[\pi_1 X]) \cong H_*(\tilde{X})$.

Prop: (essentially Hatcher Prop 4.6.1).

Given fibration $\pi: E \to B$, any path in B γ from x to y induces a homotopy equiv.

$$p_\gamma : F_x \xrightarrow{\sim} F_y,$$

well-defined up to homotopy, only depending (up to homotopy) on homotopy class of γ.

☆ **Cor:** $\{H_*(F_x)\}_{x\in B}$ w/ $F([\gamma]) := (p_\gamma)_*$ gives a local coeff. system on B.

Pf. idea: iteratively apply homotopy lifting property.

e.g., given $\gamma: I \to B$, from x to y, there's a map

$$F_x \times I \xrightarrow{g} B, \quad (f,t) \longmapsto \gamma(t)$$

along with a lift at $t=0$

$$\begin{tikzcd} & E \arrow[d] \\ F_x \arrow[ur, "\tilde{g}_0 = id_{F_x}"] \arrow[r, "g_0 = const_{\{x\}}"'] & B \end{tikzcd}$$

HLP $\Rightarrow$ get a lift

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ F_x \times I \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & B \end{tikzcd}$$

in particular $\tilde{g}_1 : F_x \longrightarrow F_{y = \gamma(1) = g_1}$.

call $p_\gamma := \tilde{g}_1$.

<u>Check</u>: independent of choices up to homotopy etc. using further H.L.P.'s.

(we've shown the desired property for <u>fibrations</u>, but can get (<u>Cor</u>)* about $\{H_*(F_x)\}_{x\in B}$ being a local coeff. system for Serre fibrations too. E.g., by CW replacement at various stages, recalling that Serre fibrations have Relative HLP for all CW pairs (X,A)).