Thm: (Leray-Serre Spectral sequence) denote fibers of $\pi: E \to B$ by $F_x := \pi^{-1}(x)$.

$\pi: E \to B$ any Serre fibration. Then, $\exists$ a spectral sequence $\{E^r_{p,q}, \partial_r\}$ defined for $r \geq 2$, with

$$E^2_{p,q} = H_p(B; \{H_q(F_x)\}_{x\in B})$$

'homology w/ coefficients in the 'local coefficient system' (bundle) of homologies of fibers.'.

(special case: if B path connected & $\pi_1(B) = 0$ or more generally local coeffi. system is trivial $\Leftrightarrow$ $\pi_1(B)$ acts trivially on $H_q(F)$ B path connected)

then $\boxed{E^2_{p,q} = H_p(B; H_q(F)) \underset{\text{UCT homology}}{\cong} H_p(B) \otimes_k H_q(F)}$,

fiber over basepoint (if over a field k or under freeness hypotheses)

converging to

$$E^\infty_{p,q} = G_p H_{p+q}(E) \quad (\text{for some filtration } F_p \text{ on } H_*(E)).$$

Pf sketch:

- "Singular method" (several ways to implement)

  In special case of a trivial fibration $E \cong F \times B$, note $C_*(E) \underset{\text{Eilenberg-Zilber thm}}{\overset{q.iso.}{\simeq}} C_*(B) \otimes C_*(F)$.

  RHS is the total cplx of a bicomplex, hence has a filtration by "degree in $C_*(B)$"; get a S.S. on $C_*(B) \otimes C_*(F)$ or - by looking at image of filtration in $C_*(E)$ - on $C_*(E)$ as desired.

  For more general $E \to B$, can similarly put a filtration on $C_*(E)$ or an analogue of $C_*(B)\otimes C_*(F)$, by restricting to chains 'adapted' to $\begin{matrix} F \to E \\ \downarrow \\ B \end{matrix}$ whose 'base degree' is $\leq p$; see [McCleary], [MIT OCW notes].

  [c.f., Dress's construction: build a homology theory for E from "singular product simplices" adapted to π:
  $$\begin{array}{ccc} \Delta^p \times \Delta^q & \longrightarrow & E \\ \downarrow \text{proj} & & \downarrow \pi \\ \Delta^p & \longrightarrow & B \end{array}.$$
  Bicomplex: has two filtrations! one filtration ⇒ page 2 = $H_p(B; \{H_q(F)\})$; other filtration ⇒ page ∞ = $H_{p+q}(E)$]

- "Cellular" method:

  Assume B is a CW complex (by CW replacement if needed). Have $\underset{\text{0-skel.}}{B^0} \subset \underset{\text{1-skel.}}{B^1} \subset B^2 \subset \cdots \subseteq B$

  $B^p :=$ p-skeleton, $C_*(E)$ chains on E.

  Let $F_p C_*(E) := C_*(E|_{B^p})$. This gives a filtration on $C_*(E)$, and hence a spectral sequence

  $$(E|_{B^0} \subseteq E|_{B^1} \subseteq \cdots) \text{ induces } (F_0 C_*(E) \subseteq F_1(C_* E) \subseteq \cdots)$$

  The associated graded chain cplx. is by definition $C_*(E|_{B^p}) / C_*(E|_{B^{p-1}}) = C_*(E|_{B^p}, E|_{B^{p-1}})$, w/ ∂_0 = relative chains differential.

  $$\Rightarrow E^1_{p,q} = H_{p+q}(E|_{B^p}, E|_{B^{p-1}}) \cong \bigoplus_{\alpha \text{ p-cells in } B} H_{p+q}(E|_{e^p_\alpha}, E|_{\partial e^p_\alpha})$$

$$\underset{\substack{E \text{ trivial} \\ \text{over } e^p_\alpha}}{\cong} \bigoplus_{\alpha \text{ p-cells}} H_p(e^p_\alpha, \partial e^p_\alpha) \otimes H_*(F_{\underset{\uparrow}{x_\alpha}})$$

some point in α.

If $\pi_1(B)$ trivial or more generally $\{H_*(F_x)\}_{x \in B}$ trivial, can identify each $H_*(F_{x_\alpha})$ w/ $H_*(F)$ by choosing any path x_α to basepoint, result independent of path, so get

$$\cong C_p^{cell}(B; H_*(F)).$$

(more generally, should identify above w/ " $C_p^{cell}(B; \{H_*(F_x)\}_{x\in B})$".)

With some additional work, can show ∂_1 on $E^1_{p,q}$ coincides w/ cellular differential (or a suitable version w/ local coefficients in general case)

$$\Rightarrow E^2_{p,q} \underset{\substack{\uparrow \\ \pi_1 B = 0 \text{ or } \{H_q(F_x)\}_{x\in B} \\ \text{trivial}}}{=} H_p^{cell}(B; H_q(F)) \cong H_p(B; H_q(F)).$$

(or more generally $= H_p^{cell}(B; \{H_q(F_x)\}_{x\in B}) \cong H_p(B; \{H_q(F_x)\}_{x\in B})$). ∎

<u>First computations</u>:

The existence of the Leray-Serre S.S. is enough to deduce a number of new computations (w/o computing explicitly ∂_r). Often if we know $H_*(B)$, $H_*(F)$, can deduce information about $H_*(E)$:

analogy: many times just having a LES in homology is enough to make new computations w/o computing explicitly connecting maps.

<u>Ex</u>: Compute $H_*(SU(4); \mathbb{Z})$.

Recall that $SU(n) = \{T \in GL(n, \mathbb{C}) \mid \overbrace{TT^* = Id}^{T \in U(n)}, \det T = 1\}$.

$SU(1) = \{Id\} = \{pt\}$ and in general observe for $n > 1$ that there's a fiber bundle (hence fibration)

$$\begin{array}{ccc} T & & SU(n) \\ \downarrow & & \downarrow \pi \\ T(\vec{e}_1) & \in & S^{2n-1} \end{array} \quad (\subseteq \mathbb{C}^n \text{ unit sphere})$$

, with fiber $\cong \pi^{-1}(\vec{e}_1) \cong SU(n-1)$. (thought of as acting on $\vec{e}_1^{\perp} \cong \mathbb{C}^{n-1}$)

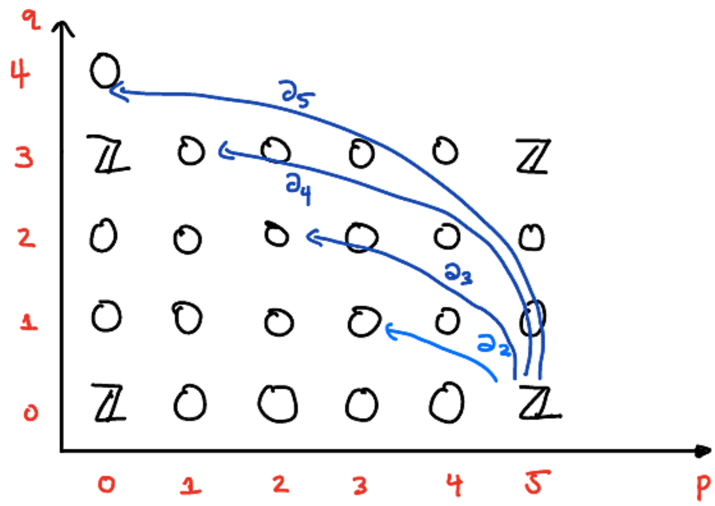
i.e., $\exists$ a fibration $SU(n-1) \to SU(n)$, $SU(n) \downarrow S^{2n-1}$, $n > 1$. $\Rightarrow$ since $SU(1) = pt$, $\boxed{SU(2) = S^3.}$

First, let's compute $H_*(SU(3))$ using $SU(2) = S^3 \to SU(3) \xrightarrow{\pi} S^5$

By Leray-Serre, (S^5 simply connected) $\exists$ a spectral sequence converging to (graded of) $H_*(SU(3))$,

w/ $E^2_{p,q} = H_p(S^5; H_q(S^3)) \underset{\substack{\text{U.C.T.}\\ \text{homology}\\ (\text{as } H_p(S^k) \text{ free})}}{\cong} H_p(S^5) \otimes_{\mathbb{Z}} H_q(S^3)$

Drawing $E^2_{p,q}$:



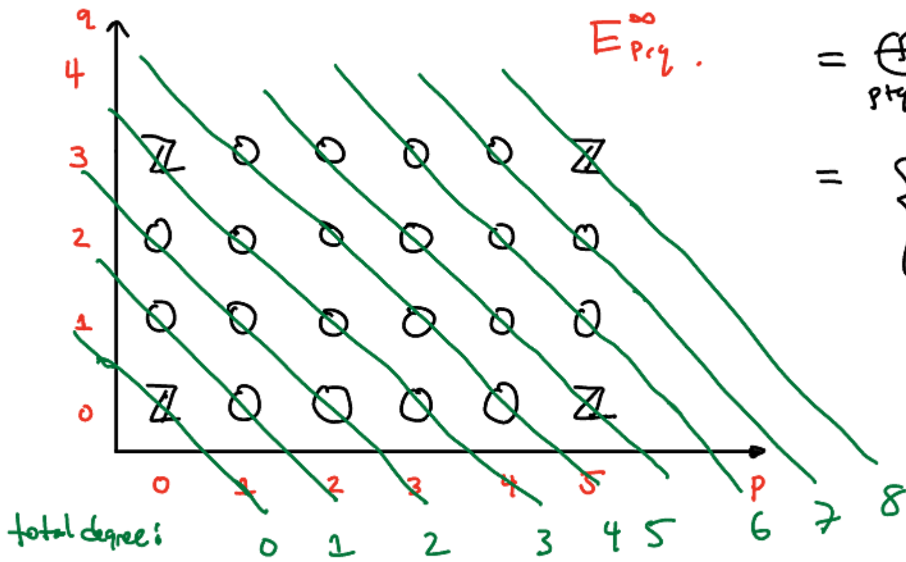
∂_r has bidegree $(-r, r-1)$

Note: $\partial_2 \equiv 0$ (as domain or codomain of each $\partial_2 = 0$), so the 4 $\mathbb{Z}$'s all survive to $E^3_{p,q}$.

In fact, inductively each further $\partial_r : E^r_{p,q} \to E^r_{p-r,\,q-r+1}$ is zero, for all r.

(to go from $\mathbb{Z}$ in position $(5,0)$ to $(0,3)$ would require bidegree $(-5,+3)$ which never happens)

$\Rightarrow$ S.S. collapses, and $E^\infty_{p,q} = G_p H_{p+q}(SU(3)) = E^2_{p,q}$. $\Rightarrow \forall k$ $\bigoplus_p G_p H_k(SU(3))$



$= \bigoplus_{p+q=k} E^\infty_{p,q} = \bigoplus_{p+q=k} E^2_{p,q}$

$= \begin{cases} \mathbb{Z} & k = 0,3,5,8 \\ 0 & \text{else.} \end{cases}$

Since each group is free, no extension issues $\Rightarrow$ $H_*(SU(3);\mathbb{Z}) \cong H_*(S^3) \otimes H_*(S^5) = \begin{cases} \mathbb{Z} & k=0,3,5,8 \\ 0 & \text{else.} \end{cases}$

$\cong H_*(S^3 \times S^5)$.

Now, $SU(3) \to SU(4)$ induces a S.S. w/ $E^2_{p,q} \cong H_p(S^7; H_q(SU(3)))$

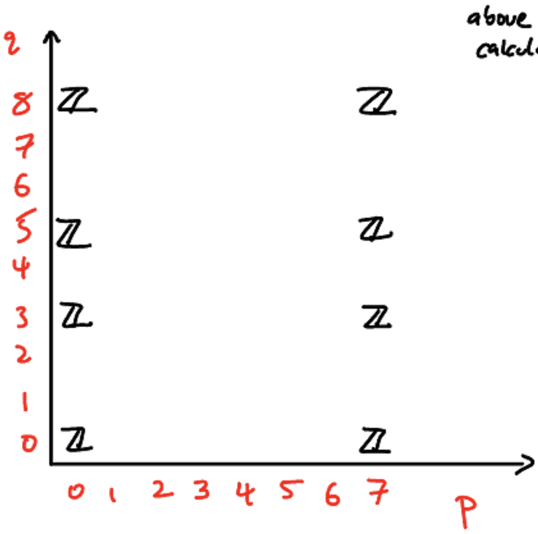
$\qquad\qquad\quad \downarrow$

$\qquad\quad\;\; S^7$

$$\cong H_p(S^7) \otimes H_q(S^3 \times S^5)$$

(UCT + above calculation)

A rough picture of $E^2_{p,q}$ (only showing non-zero groups).



Again inductively for every r, supposing $E^r_{p,q}$ looks like the above, note there is no (p,q) s.t. the domain and codomain of ∂_r on $E^r_{p,q}$ are both non-zero for degree reasons $\Rightarrow \partial_r \equiv 0 \;\; \forall\, r \geq 2$. $\Rightarrow$ collapse at $E^2_{p,q}$.

$$\Rightarrow H_*(SU(4)) \underset{\substack{\text{again}\\ \text{no extension}\\ \text{issues}}}{\cong} H_*(S^7) \otimes H_*(S^3 \times S^5) \cong H_*(S^3 \times S^5 \times S^7) = \begin{cases} \mathbb{Z} & \deg 0, 3, 5, 7, 8, 10, 12, 15 \\ 0 & \text{else.} \end{cases}$$

Rmk: It turns out $H_*(SU(n); \mathbb{Z}) \cong H_*(S^3 \times S^5 \times \cdots \times S^{2n-1}; \mathbb{Z})$ even though when $n > 2$, these spaces aren't homeomorphic (in fact have different homotopy groups, so not htpy equivalent).

$\boxed{\text{Ex}}$: Compute $H_*(\Omega S^n)$ for $n > 1$. (for $n = 1$, note $\Omega S^1 \cong \mathbb{Z}$, a discrete space, by covering space arguments).
↑ based loop space (fix a basepoint $*$).

This computation is a nice example of how sometimes, from known information about $H_*(E)$ & $H_*(B)$ we can learn about $H_*(F)$.

We will use the Serre fibration $\Omega S^n \to PS^n$ ← paths $\gamma: I \to S^n$ w/ $\gamma(0) = *$ (no constraint on $\gamma(1)$)

$$\begin{array}{c} PS^n \\ \downarrow \pi \\ S^n \end{array} \qquad \pi: \gamma \longmapsto \gamma(1).$$

(means have a Serre fibration $PS^n \to S^n$, fibers all htpy equiv. to ΩS^n; observe $\pi^{-1}(*) = \Omega S^n$).

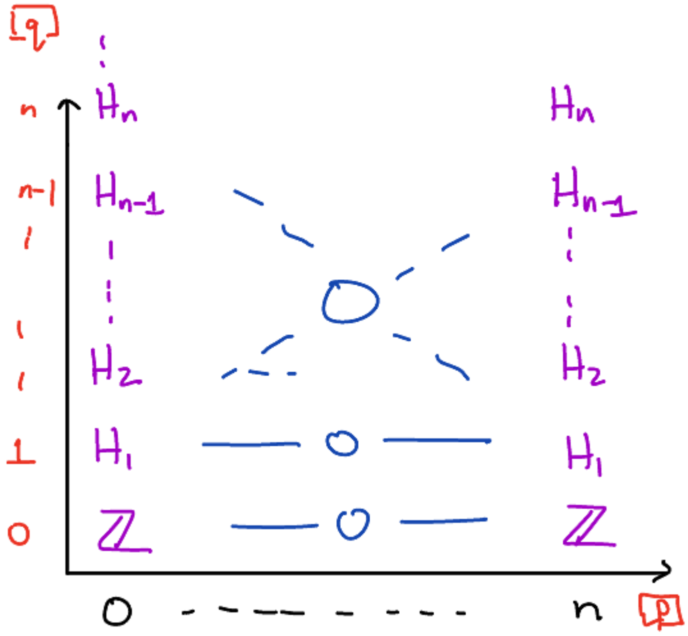
Fact: The total space PS^n is contractible. (exercise). $\Rightarrow H_i(PS^n) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{else.} \end{cases}$

$\Rightarrow$ in the Leray-Serre spectral sequence, we must have $E^\infty_{p,q} = \begin{cases} \mathbb{Z} & (p,q)=(0,0) \\ 0 & \text{else.} \end{cases}$

On the other hand, denoting by $H_i := H_i(\Omega S^n)$, we know
unknown quantity of interest.

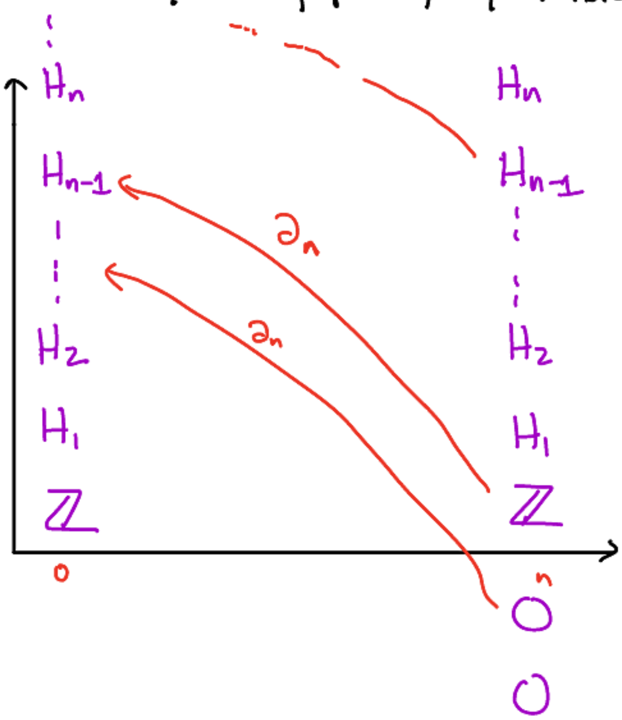
$E^2_{p,q} = H_p(S^n) \otimes H_q(\Omega S^n) = H_p(S^n) \otimes H_q$.

Picture of $E^2_{p,q}$:



All groups aside from $E^2_{0,0}$ must be killed by some ∂_r, given we know $E^\infty_{p,q} = \begin{cases} \mathbb{Z} & (p,q)=(0,0) \\ 0 & \text{else.} \end{cases}$

On the other hand, the only possible non-zero differential on this picture, on any page of S.S., is ∂_n, as it lowers the degree of p precisely by n. Hence, ∂_n must be an isomorphism, except when domain or codomain is $E^n_{0,0} = \mathbb{Z}$.



Note: for $1 \leq i \leq n-2$,

$\partial_n : \underset{\substack{\parallel \\ 0}}{E^n_{n,i-n+1}} \xrightarrow{\cong} E^n_{0,i} = H_i$

(b/c $i-n+1<0$).

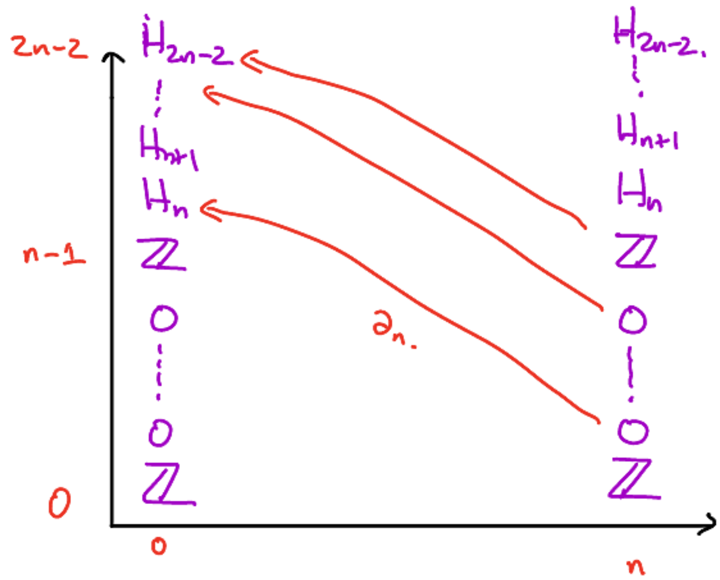
$\Rightarrow$ $\boxed{H_i(\Omega S^n) = 0 \text{ for } 1 \leq i \leq n-2.}$

Next, when $i = n-1$,

$$\partial_n : \quad \mathbb{Z} = E^n_{n,0} \xrightarrow{\cong} E^n_{0,n-1} = H_{n-1}$$

$$\Rightarrow \boxed{H_{n-1}(\Omega S^n) \cong \mathbb{Z}.}$$

So the picture looks like this:



$\Omega S^1 = \mathbb{Z}$

$\cdots \bullet \quad \bullet \quad \bullet \quad \bullet \cdots$

$H_0(\Omega S^1) = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$

exercise: see this using L-S.S.S. w/ page $2 :=$ twisted coefficients.

$\Rightarrow$ we see $H_n = 0$, $H_{n+1} = 0$, ---, $H_{2n-3} = 0$, $H_{2n-2} = \mathbb{Z}$.

Inductively repeating get

$$H_i(\Omega S^n) \cong \begin{cases} \mathbb{Z} & i = k(n-1) \quad k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Naturality of spectral sequences:

General algebraic setting: $(C_*, F_p C_*)$, $(C_*', F_p' C_*')$ two filtered chain complexes.
A map $f : C_* \to C_*'$ is a filtered chain map if chain map & $f(F_p C_*) \subset F_p' C_*'$ for all p.

Lem/Exercise: check from definitions that such an f induces maps

$$f_*^r : \underbrace{E^r_{p,q}}_{\text{s.s. for } C_*} \longrightarrow \underbrace{E'^r_{p,q}}_{\text{s.s. for } C_*'}, \quad \text{chain map for } \partial_r \quad (\text{i.e., } f_*^r \circ \partial_r = \partial_r \circ f_*^r),$$

s.t., f_*^{r+1} on $E^{r+1}_{p,q}$ is the map on homology induced by f_*^r.

(call such f_*^r a morphism of spectral sequences).

e.g., $f_*^0 = (Gf): G_p C_{p+q} \longrightarrow G'_p C'_{p+q}$ (well-defined b/c f sends F_p to F_p', F_{p-1} to F'_{p-1}).

Cor: If any f_*^r is an isomorphism (for all p,q), (& if filtrations bdd) then f induces a homology isomorphism $f_*: H_*(C_*) \xrightarrow{\cong} H_*(C'_*)$.

e.g., if $f_2^*: H_{p+q}(G_p(C_*)) \xrightarrow{\cong} H_{p+q}(G_p' C_*')$.

Pf: If so, then $f_\infty^* = G_p[f]: G_p H_{p+q}(C_*) \xrightarrow{\cong} G_p H_{p+q}(C_*)$ for all p, q.

But $\exists$ a commutative diagram of SES's.

$$\begin{tikzcd}
0 \arrow[r] & F_{p-1}H_{p+q}(C_*) \arrow[r] \arrow[d, "{[f]|_{F_{p-1}}}"] & F_p H_{p+q}(C_*) \arrow[r] \arrow[d, "{[f]|_{F_p}}"] & G_p H_{p+q}(C_*) \arrow[r] \arrow[d, "{G_p[f]}"] & 0 \\
0 \arrow[r] & F_{p-1}H_{p+q}(C_*) \arrow[r] & F_p H_{p+q}(C_*) \arrow[r] & G_p H_{p+q}(C_*) \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ inductively using 5-Lemma (using F_p bounded) get that $[f]: H_*(C_*) \xrightarrow{\cong} H_*(C'_*)$. $\boxtimes$.

Turning to fibrations, if we have a map of fibrations, in the sense of

$$\begin{tikzcd}
F \arrow[r, "f_F"] \arrow[d] & F' \arrow[d] \\
E \arrow[r, "f_E"'] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\
B \arrow[r, "f_B"'] & B'
\end{tikzcd}$$

then Lem: Such a map between fibrations induces a morphism between the associated Leray-Serre spectral sequences, which on page 2 agrees with the map

$$H_*(B, \{H_*(F_x)\}_{x\in B}) \longrightarrow H_*(B', \{H_*(F'_{x'})\}_{x'\in B'})$$

induced by f_B and $\{f_{F_x}\}_{x\in B}$.

(both B, B' simply connected or associated local coefficient systems trivial: the map $H_*(B; H_*(F)) \to H_*(B'; H_*(F'))$ induced by f_F and f_B).

Idea of proof: If B, B' CW cplxes (as used in construction of spectral sequence above), can replace f_B up to htpy w/ a cellular map i.e., it respects skeletal filtration of B, B' hence induces a filtered chain map from $C_*(E)$ to $C_*(E')$. Now check on page 2 ... $\boxtimes$

Using this, can prove a sort of '5-lemma' for maps between fibrations:

<u>Prop</u> (Hatcher-SS, Prop 1.12): Say have a map of fibrations as above, & let's just assume $\pi_1 B = \pi_1 B' = \{*\}$ OR local coeff. systems trivial in both cases (for simplicity). Then, if two out of the three maps $F \xrightarrow[f_F]{} F'$, $E \xrightarrow[f_E]{} E'$, $B \xrightarrow[f_B]{} B'$ induce homology isos for $H_*(-;R)$ PID (so UCT for chain cplxes$_R$ applies), then so does the third.

<u>Pf</u>:
Simplest case: say $(f_F)_*$ & $(f_B)_*$ are isos. A map of fibrations induces a morphism of L-S spectral sequences, hypotheses $\Rightarrow$ $f_2^*: E^2_{p,q} \xrightarrow{\cong} E^{2\,\prime}_{p,q}$. By alg. corollary above, this implies f^*_∞, and hence $(f_E)_*$ are isomorphisms.

one of the other two cases spelled out in [Hatcher-S.S.] $\boxtimes$.

<u>Rmk</u>: Using morphisms of spectral sequences associated to maps of fibrations can help compute differentials, see e.g., [McCleary, Example 5.H].

<u>Cohomological spectral sequences and products</u>

- A <u>cohomological spectral sequence</u> is defined in basically the same way but with all arrows reversed. i.e., have R-modules $\{E_r^{p,q}\}$, defined for all $r \geq r_0$ & differentials $\delta^r: E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1}$, with $E_{r+1} = H^\cdot(E_r, \delta_r)$.

- A co-chain complex (C^*, δ) w/ a <u>decreasing filtration</u> $F_p C_* \supseteq F_{p+1} C_* \supseteq \cdots$ of co-chain complexes gives a spectral sequence with

$$E_r^{p,q} = \frac{\{x \in F_p C^{p+q} \mid \delta x \in F_{p+r} C^{p+q+1}\}}{F_{p+1} C^{p+q} + \delta(F_{p-r+1} C^{p+q-1})}$$

again $\frac{A}{B} := \frac{A}{A \cap B}$.

(again $E_1^{p,q} = H^{p+q}(G_p C^*)$, where G_p means F_p / F_{p+1}), converging (if filtration bounded) to $G_p H^{p+q}(C_*)$.

We can now also consider <u>products</u>. Suppose the filtered cochain complex $(C^*, F_p C^*)$ is equipped with a product:

$$\star : C^i \times C^j \longrightarrow C^{i+j},$$

such that

deg(α)

- $\star$ satisfies Leibniz rule, i.e., if $\alpha \in C^i$, $\beta \in C^j$ then $\delta(\alpha \star \beta) = \delta\alpha \star \beta + (-1)^i \alpha \star \delta\beta$

  ($\Rightarrow \star$ descends to cohomology)

- $\star$ is filtered i.e., it respects the filtration, in the sense that

$$F_p C^* \otimes F_{p'} C^* \xrightarrow{\star} F_{p+p'} C^* .$$

Then $\star$ induces a well-defined product on associated graded complexes (also satisfying Leibniz):

$$\star_0 : \frac{F_p C^*}{F_{p+1} C^*} \otimes \frac{F_{p'} C^*}{F_{p'+1} C^*} \longrightarrow \frac{F_{p+p'} C^*}{F_{p+p'+1} C^*}$$

More generally, it's easy to see that $\star$ induces a well-defined map

$$\star_r : E_r^{p,q} \otimes E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$

sending $\quad [x] \otimes [y] \longmapsto [x \star y]$.

Prop: Given a $\star$ as above, the induced products satisfy:

where $\alpha \in E_r^{p,q}$

- δ_r is a derivation w.r.t. $\star_r$: $\delta_r(\alpha \star_r \beta) = (\delta_r \alpha) \star_r \beta + (-1)^{p+q} \alpha \star_r \delta_r(\beta)$
- $\star_{r+1}$ is the product on cohomology (of E_r, δ_r) induced by $\star_r$.

$\boxed{\text{Call } \{E_r^{p,q}, \delta_r, \star_r\} \text{ a spectral sequence of algebras (or multiplicative S.S.)}}$

- If filtration bounded (so S.S. converges), the limiting product

$$\star_\infty : G_p H^i \otimes G_{p'} H^j \longrightarrow G_{p+p'} H^{i+j}$$

is the top associated graded piece of the product $[\star] : F_p H^i \otimes F_{p'} H^j \longrightarrow F_{p+p'} H^{i+j}$.

Pf: exercise.

Warning: even if $\nexists$ extension problems, so $\{G_p H^{p+q}\}_{p,q}$ determines $\{H^k\}_k$, $\star_\infty$ may not determine $\star$! (examples: Hatcher-S.S. Ex. 1.17, McCleary Ex. 1.J). Here's a very special case when it always does:

Lem ([McCleary Ex 1.K]): If $E_\infty^{*,*}, \star_\infty \cong G_* H^*, G_* \star$ is a free, graded commutative, bigraded algebra, then

$$H^* \cong \bigoplus_* \bigoplus_{i+j=*} E_\infty^{i,j} \quad \text{as algebras.}$$

for $x \in E_\infty^{p,q}$, $y \in E_\infty^{r,s}$
$x \cdot y = (-1)^{(r+s)(p+q)} y \cdot x$

N.B. As before, this construction is suitably natural in $(F_p C_*, \star)$.

Cohomological Leray–Serre spectral sequence

$F_p C_*$ chain cplx w/ increasing (bounded) filtration. Then dualizing $C^* := \mathrm{Hom}(C_*, R)$, the dual co-chain cplx inherits a (bounded) decreasing filtration:

$$F_p \operatorname{Hom}(C_*, R) := \operatorname{Ann}(F_{p-1} C_*) = \{\phi : C_* \to R \mid \phi|_{F_{p-1}C_*} \equiv 0\}$$

If $F_{p-1} C_*$ is free inside $F_p C_*$, then

$$G_p \operatorname{Hom}(C_*, R) = \frac{\operatorname{Ann}(F_{p-1} C_*)}{\operatorname{Ann}(F_p C_*)} \cong \operatorname{Hom}(G_p C_*, R).$$

$\Rightarrow$ obtain a cohomological spectral sequence converging to $G_p H^{p+q}(\operatorname{Hom}(C_*, R))$ with

$$E_1^{p,q} = H^{p+q}(\operatorname{Hom}(G_p C_*, R)),$$

& δ_1 induced by applying $\operatorname{Hom}(-,R)$ to the differential ∂_1 on $E^1_{p,q}$ page of the homological S.S. for C_*.

$\pi : E \to B$ Serre fibration, then applying $\operatorname{Hom}(-,R)$ to the filtration on $C_*(E)$ inducing homological Leray-Serre S.S. gives a cohomological version of the Leray-Serre spectral sequence

$$E_2^{p,q} = \underbrace{H^p(B; \{H^q(F_x; R)\}_{x \in B})}_{(*)} \quad \text{converging to } E_\infty^{p,q} = G_p H^{p+q}(E; R)$$

Can check: the cup product $\cup$ on $C^*(E;R)$ respects the filtration.

$\Rightarrow$ get a spectral sequence of algebras $(E_r^{p,q}, \delta_r, \star_r)$ (i.e., $\star_r$ derivation, $H^*(\star_r) = \star_{r+1}$)

What is $\star_2$ on $E_2^{p,q}$? Can be described in terms of the 'usual' cup product on $(*)$.

For simplicity, let's assume the local coefficient system $\{H^*(F_x; R)\}_{x \in B}$ is trivial (e.g., if $\pi_1 B = \{*\}$, B path conn.) $(**)$ and working over a field.

$(**) \Rightarrow$ $E_2^{p,q} = H^p(B) \otimes H^q(F)$. In this case for α of bidegree (p,q), β of bidegree (p',q'),

$$\boxed{\alpha \star_2 \beta := (-1)^{qp'} \alpha \cup \beta} \in H^{p+p'}(B) \otimes H^{q+q'}(F)$$

means, cup in B and cup in F.

"Koszul sign for moving the $H^q(F)$ part of α tensor past $H^{p'}(B)$ part of β tensor to cup."

In general: For any local coeff. systems $\mathcal{G}, \mathcal{G}'$, $\exists$ cup product $H^*(B; \mathcal{G}) \otimes H^*(B; \mathcal{G}') \to H^*(B, \mathcal{G} \otimes \mathcal{G}')$

if $\mathcal{G} = \{H^q(F_x)\}_{x \in B}$, $\mathcal{G}' = \{H^{q'}(F_x)\}_{x \in B}$ applying above canonical cup product plus fiberwise cup product $H^q(F_x) \otimes H^{q'}(F_x) \to H^{q+q'}(F_x)$ gives

$\cup : H^p(B; \{H^q(F_x)\}_{x \in B}) \otimes H^{p'}(B; \{H^{q'}(F_x)\}_{x \in B}) \longrightarrow H^{p+p'}(B; \{H^q(F_x)\}_{x \in B})$, which coincides w/ $\star_2$ up to the same sign $(-1)^{qp'}$.

In some cases, knowing there's a product structure drastically simplifies computations using the Leray-Serre spectral sequence. Idea: the fact that $\star_r$ is a derivation determines $\delta_r(\underbrace{\alpha \star_r \beta}_{\text{higher degree}})$ in terms of $\delta_r\alpha$, $\delta_r\beta$, $\star_r, \alpha, \beta$.

$\Rightarrow$ can compute more of δ_r!

Also, can use it to compute ring structures (of $H^*(F)$ or $H^*(B)$, or $H^*(E)$ given caveats/warnings above).

Examples:

Ex: Compute $H^*(U(n);\mathbb{Z})$ as a ring.

Claim: $H^*(U(n);\mathbb{Z}) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}]$ w/ $|x_{2k-1}| = 2k-1$. as rings.

means exterior algebra on $x_1, x_3, \ldots$ i.e., $x_{2k-1}x_{2l-1} = -x_{2l-1}x_{2k-1}$

$(\mathbb{Z}\langle x_1, x_3, \ldots, x_{2n-1}\rangle / x_i^2 = 0,\ x_i x_j = -x_j x_i)$

$(x_{2k-1})^2 = 0$.

note: equivalently, this is the free graded commutative algebra over $\mathbb{Z}$ on $x_1, x_3, \ldots, x_{2k-1}$. Graded commutativity $\Rightarrow (x_{2k-1})^2 = -(x_{2k-1})^2 \Rightarrow x_{2k-1}^2 = 0$, etc.

$x \cdot y = (-1)^{|y||x|} y \cdot x$.

(i.e., $H^*(U(n);\mathbb{Z}) \cong H^*(S^1) \otimes H^*(S^3) \otimes \cdots \otimes H^*(S^{2n-1}) \cong H^*(S^1 \times S^3 \times \cdots \times S^{2n-1})$ as rings).

Pf: $n=1$: $U(1) = S^1$ so result is true.

Inductively, assume true for $n-1$. $\exists$ a fibration

$$\begin{array}{ccc} U(n-1) \longrightarrow U(n) & & T \\ \downarrow & & \downarrow \\ S^{2n-1} & & T(\vec{e}_1) \end{array}$$

unit sphere in $\mathbb{C}^n$. (Simply connected as $n>1$).

$\Rightarrow$ get L-S. s.s. of algebras with

$$(E_2, \star_2) = H^*(S^{2n-1}; H^*(U(n-1))) \underset{\text{U.C.T.}}{\cong} H^*(S^{2n-1}) \otimes H^*(U(n-1)), \cup$$

cup product.

total degree k part is $\bigoplus_{i+j=k} E_2^{i,j}$

$$\underset{\substack{\text{induction}\\ +\text{Künneth}}}{\cong} H^*(S^1 \times S^3 \times \cdots \times S^{2n-1}) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}].$$

as graded rings, using total degree on E_2, & usual degree on RHS.

Now, note that δ_2 (and more generally each δ_r) increases total degree by 1 (bidegree $(r, -(r-1))$).
Since every non-unit element in E_2 (and inductively E_r) is in odd degree,

$\Rightarrow$ $\delta_r \equiv 0$ for all $r \geq 2$ $\Rightarrow$ collapse at E_2..

Hence $(E_\infty, \star_\infty) \cong \Lambda[x_1, x_3, \dots, x_{2n-1}]$. $\cong \left(\bigoplus_p G_p H^*(U(n)), G\smile\right)$

This is a free (bigraded) graded commutative algebra $\Longrightarrow$ $\boxed{H^*(U(n)) \cong \Lambda[x_1, \dots, x_{2n-1}]}$
if one tracks (p,q)'s.
Remark above
as rings.
[cf. McCleary ex 1.K]

(Done on 4/28)

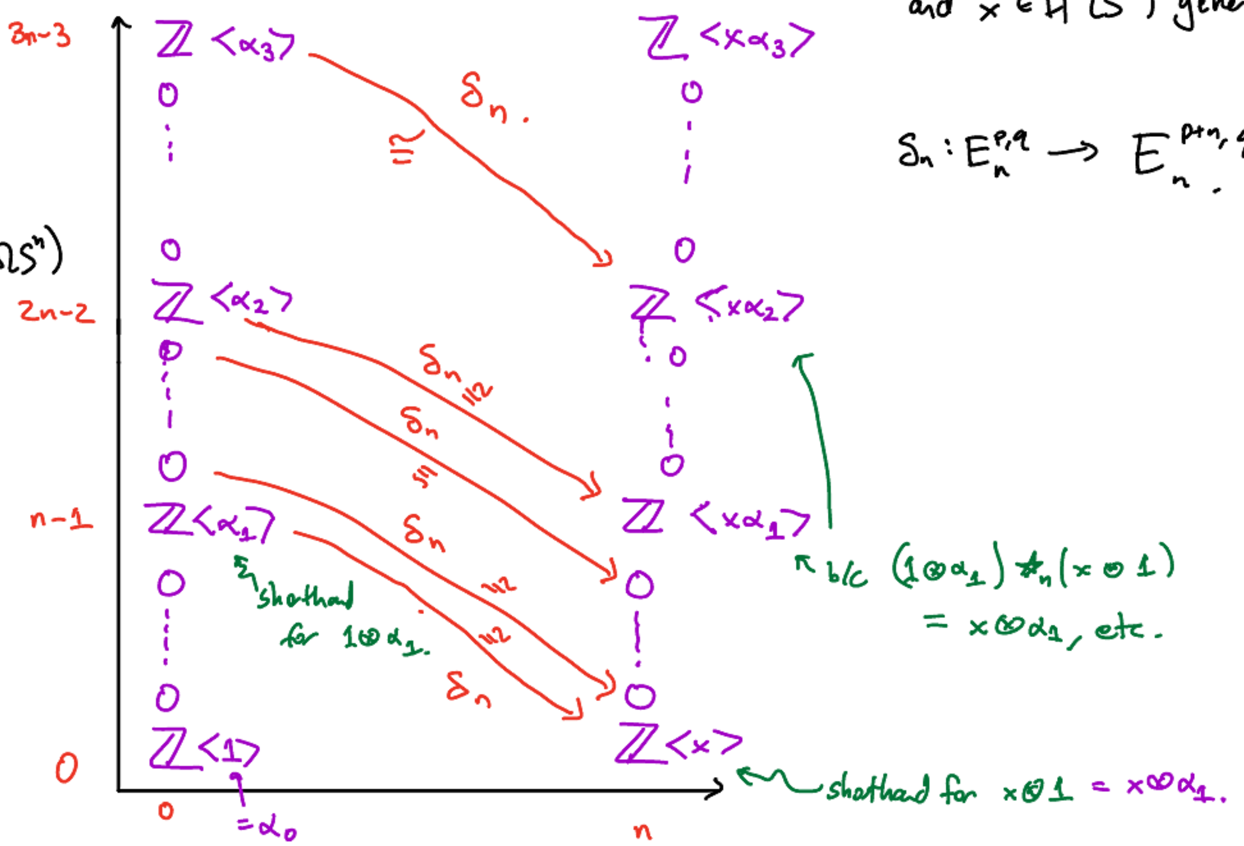
Ex: compute $H^*(\Omega S^n; \mathbb{Z})$ as a ring, $n > 1$.

We revisit the path-space loop-space fibration: $\Omega S^n \to PS^n \downarrow S^n$ (contractible), look at the Leray-Serre S.S. again, this time for cohomology:

The same arguments as before imply $H^j(\Omega S^n) = \begin{cases} \mathbb{Z} & j = k(n-1) \\ 0 & \text{else.} \end{cases}$, & n^{th} page looks like:

with all δ_n (in picture.) isomorphisms (except one to/from $E_n^{0,0}$). Call α_k the generator of $H^{k(n-1)}(\Omega S^n)$, ($\alpha_0 = 1$) and $x \in H^n(S^n)$ generator.

$E_n^{p,q} \cong H^p(S^n) \otimes H^q(\Omega S^n)$



$\delta_n : E_n^{p,q} \to E_n^{p+n,\, q-n+1}$.

Can WLOG choose generators x, α_i s.t. $\delta_n : \alpha_k \longmapsto x\alpha_{k-1}$. ($\delta_n : \alpha_1 \longmapsto x$)

Note: by derivation property of $\star_n$, we learn that

$$\delta_n(\alpha_1^2) = \delta_n(\alpha_1 \star_n \alpha_1) = (-1)^{n-1}\alpha_1 \delta_n \alpha_1 + (\delta_n \alpha_1)\alpha_1.$$

$$= \begin{cases} 0 & n \text{ even} \\ 2x\alpha_1 & n \text{ odd} \end{cases}$$

(know: $\delta_n \cong$ and $\delta_n : \alpha_2 \longmapsto x\alpha_1$) gen. for grp in which α_1^2 lives

We deduce that (as $\delta_n \cong$.) $\alpha_1^2 = 2\alpha_2$ $\underline{n \text{ odd}}$, and $\alpha_1^2 = 0$ $\underline{n \text{ even}}$.

More generally,

$$\delta_n(\alpha_1\alpha_k) = \delta_n(\alpha_1)\alpha_k + (-1)^{n-1}\alpha_1\delta_n\alpha_k$$
$$= x\alpha_k + (-1)^{n-1}\alpha_1 x \alpha_{k-1}.$$

$$\delta_n(\alpha_k\alpha_\ell) = \delta_n(\alpha_k)\alpha_\ell + (-1)^{k(n-1)}\alpha_k\delta_n(\alpha_\ell)$$
$$= x\alpha_{k-1}\alpha_\ell + (-1)^{k(n-1)} x \alpha_k\alpha_{\ell-1}.$$

$$\Rightarrow \alpha_k\alpha_\ell = \text{preimage under } \delta_n \text{ of } \begin{cases} x(\alpha_{k-1}\alpha_\ell + \alpha_k\alpha_{\ell-1}) & k \text{ even or } n \text{ odd.} \\ x(\alpha_{k-1}\alpha_\ell - \alpha_k\alpha_{\ell-1}) & k \text{ odd } \underline{\text{and}} \ n \text{ even.} \end{cases}$$

$\boxed{n \text{ odd}}$:

$$\alpha_1\alpha_2 = \delta_n^{-1}(x(\alpha_2+\alpha_1^2)) = \delta_n^{-1}(3x\alpha_2) = 3\alpha_3$$
$$\Rightarrow \alpha_1^3 = 2\alpha_1\alpha_2 = 3!\alpha_3.$$

Inductively, one sees $\alpha_1^k = k!\alpha_k$, i.e., $H^\bullet(\Omega S^n) = \mathbb{Z}[\overset{\deg.(n-1)}{\alpha}, \frac{1}{2}\alpha^2, \frac{1}{3!}\alpha^3, \dots, \frac{1}{k!}\alpha^k, \_\_]$

$= \Gamma_{\mathbb{Z}}(\alpha)$ is a $\underline{\text{divided power algebra}}$.

(rationally, just polynomials in α).

$\boxed{n \text{ even}}$:

$\boxed{\alpha_2^k = k!\alpha_{2k}}$ so $\alpha_2, \alpha_4, \alpha_6, \dots$ $\underline{\text{generate a divided power algebra}}$ $\Gamma_{\mathbb{Z}}[\beta]$ $|\beta| = 2n-2$.

$\boxed{\alpha_1^2 = 0}$ $\delta(\alpha_1\alpha_2) = x(\alpha_2 - \alpha_1^2) = x\alpha_2 \Rightarrow \boxed{\alpha_1\alpha_2 = \alpha_3}$

Inductively, can similarly show $\alpha_1\alpha_{2k-1} = 0$ and $\boxed{\alpha_1\alpha_{2k} = \alpha_{2k+1}}$

$$\Rightarrow H^\bullet(\Omega S^n) \cong \Gamma_{\mathbb{Z}}[\beta] \otimes \Lambda[\alpha]$$

$\beta = \alpha_2$, deg $2(n-1)$ $\alpha = \alpha_1$, deg $n-1$. (e.g., $\alpha_{2k+1} = \alpha_1\alpha_{2k}$
$= \alpha_1 \cdot \frac{1}{k!}\alpha_2^k$
$= \alpha \cdot \frac{1}{k!}\beta^k$).

(on 4/26)

Example: Let's recompute $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ as a ring using L-S. S.S. Denote $H^p := H^p(\mathbb{CP}^\infty)$ unknown. $[H^0 = \mathbb{Z}]$

The fibration $S^1 \to S^{2k+1}$ ← unit sphere in $\mathbb{C}^{k+1}$, $S^{2k+1} \downarrow \mathbb{CP}^k$ induces (taking $k\to\infty$) a fibration $S^1 \to S^\infty$, $S^\infty \downarrow \mathbb{CP}^\infty$.

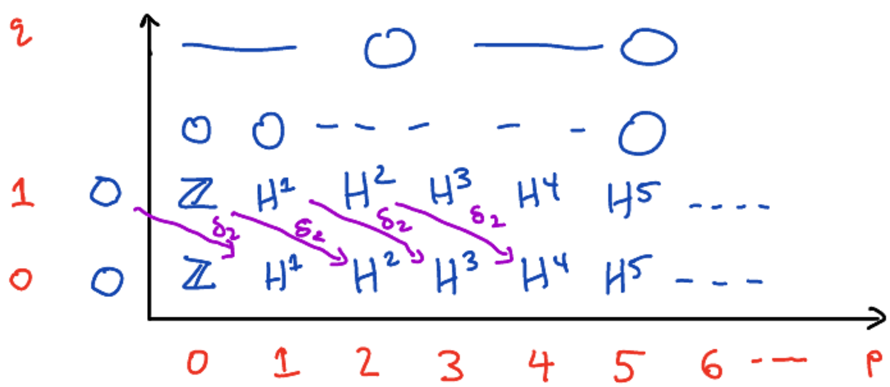
$\pi_1(\mathbb{CP}^\infty) = \{*\}$, and S^∞ contractible $\Rightarrow$ the associated L-S. S.S. has

$$E_2^{p,q} = H^p(\mathbb{CP}^\infty; H^q(S^1)) \cong \begin{cases} H^p & \text{when } q = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

w/ $\bigoplus_{i+j=k} E_\infty^{i,j} = \bigoplus_p G_p H_k(S^\infty) = \begin{cases} \mathbb{Z} & k=0 \\ 0 & \text{else}, \end{cases}$

i.e., $E_\infty^{p,q} = \mathbb{Z}$ if $(p,q) = (0,0)$ and 0 otherwise.

Picture: of $E^2_{p,q}$:



$(H^0 = \mathbb{Z})$.

Since $E_\infty^{p,q} = 0$ except at $(0,0)$, every other grp must eventually become zero.

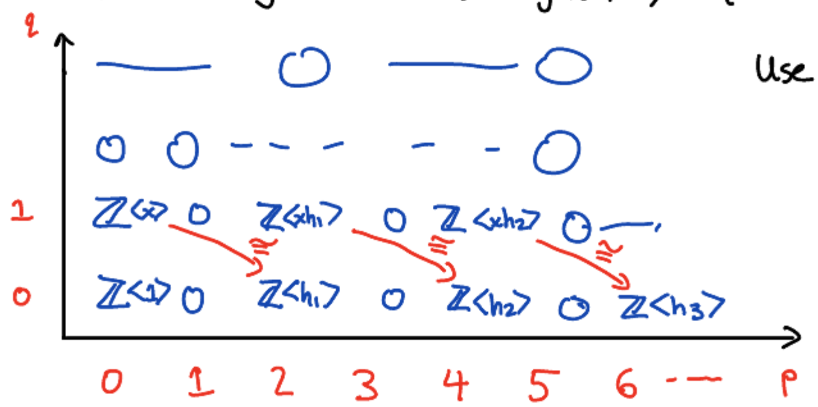
The only page with possibly non-zero differential is page 2 (every other δ_r, $r>2$, decreases q degree by >1 so has either domain or codomain 0).

$\Rightarrow$ must have $\delta_2 : E_2^{p,1} \xrightarrow{\cong} E_2^{p+2,0}$ provided $p \neq -2$;

So, $H^1 = 0$, $H^1 \cong H^3 = 0$, & inductively $H^{2k+1} = 0$. $\forall k$.

and $H^0 \cong H^2$, so $H^2 \cong \mathbb{Z}$, $H^2 \cong H^4$ so $H^4 \cong \mathbb{Z}$, inductively $H^{2k} = \mathbb{Z}$ $\forall k$

<u>Algebra structures</u>: Denoting $x \in H^1(S^1)$ a generator, $h_i \in H^{2i}$ generator (w/ $h_0 = 1$):



Use shorthand $1 := 1 \otimes 1$.

x for $x \otimes 1$

h_i for $1 \otimes h_i$

xh_i for $x \otimes h_i$ (note $xh_i := (x \otimes 1) \smile_2 (1 \otimes h_i)$)

WLOG, inductively after choosing h_i, choose h_{i+1} s.t. $\underline{\delta_2(xh_i) = h_{i+1}}$. (*)

(Leibniz rule tells us info about a higher degree $\delta_2(ab)$ in terms of $\delta_2(a)$, $\delta_2(b)$, a, b, products)

Leibniz rule for the algebra structure on $E_2^{p,q}$ $\Rightarrow$

$$\delta_2(xh_1) = \delta_2(x)h_1 + x\underbrace{\delta_2(h_1)}_{0} \quad \text{(using } \star_2 : ab \text{ means } a \star_2 b\text{)}.$$
$$= h_1 \cdot h_1 = h_1^2.$$

On the other hand, $\delta_2(xh_1) = h_2$ $\Rightarrow$ $h_1^2 = h_2$. Inductively, assuming $h_1^k = h_k$, we discover

$$\underset{\substack{\| \\ h_{k+1}}}{\delta_2(xh_k)} \overset{(*)}{} \underset{\text{inductively}}{=} \delta_2(xh_1^k) \underset{\substack{\text{Leibniz} \\ \& \ \delta_2(h_1)=0}}{=} h_1^{k+1}.$$

$\Rightarrow h_1^{k+1} = h_{k+1}$. So inductively $h_1^m = h_m$ for all m, and $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[h]$, $|h| = +2$. (h = h_1 above) ∎

— 4/28/2021 —

Additional properties of, and applications of, the Leray-Serre Spectral Sequence

Let's return to the (homological) Leray-Serre spectral sequence. A few groups (& maps between them) in the spectral sequence have particularly clear significance in terms of the original fibration.

For what follows $\boxed{\text{assume have a fibration } F \to E \xrightarrow{\pi} B \text{ w/ trivial local coefficient system}}$ $\{H_q(F_x)\}_{x \in B}$ (e.g., if B path connected & $\pi_1(B) = \{*\}$).

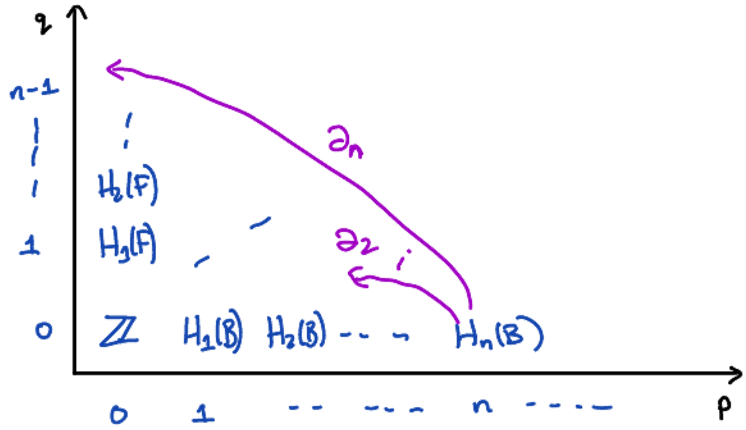
$\Rightarrow$ L-S. S.S. for $H_*(E)$ has $\boxed{E^2_{p,q} \cong H_p(B, H_q(F))}$

- Edge homomorphisms

$\boxed{\text{Further assume fibers } F \text{ are connected, as is } B.}$

Note the bottom edge of E^2 is canonically $\cong H_*(B)$ $\left(E^2_{p,0} := H_p(B, H_0(F)) \cong H_p(B) \right)$

(similarly left vertical edge is canonically $\cong H_*(F)$ — $(E^2_{0q} = H_0(B, H_q(F)) \cong H_q(F))$..



Being at the bottom of a first quadrant spectral sequence, no differentials can hit any $E^r_{n,0}$.

$$\Rightarrow E^{r+1}_{n,0} = \ker\left(\partial_r : E^r_{p,0} \to E^r_{p-r,r-1}\right) \subseteq E^r_{n,0}.$$

And the last page k for which ∂_k can be non-zero on $E^k_{n,0}$ is $k=n$, as $\partial_{n+1} : E^{n+1}_{n,0} \to E_{-1,n} \equiv 0$ (first quadrant S.S.).

So $E^{n+1}_{n,0} = E^\infty_{n,0}$, and we have a chain of inclusions

$$G_n H_n(E) = E^\infty_{n,0} = E^{n+1}_{n,0} \subseteq E^n_{n,0} \subseteq E^{n-1}_{n,0} \subseteq \cdots \subseteq E^2_{n,0} = H_n(B).$$

Now observe that $G_s H_{n=s+(n-s)}(E) = E^\infty_{s,n-s} \equiv 0$ for $s>n$.

$\Rightarrow H_n(E) \cong F_n H_n(E)$. (an explanation for this is that all of $H_n(E)$ is seen by $E|_{B^n} \xhookrightarrow{\text{incl.}} E$; the image in homology is precisely $F_n H_n(E)$, which is therefore all of $H_n(E)$).

n-skeleton

Hence there is a projection map

$$H_n(E) = F_n H_n(E) \twoheadrightarrow F_n H_n(E) / F_{n-1} H_n(E) = G_n H_n(E) = E^\infty_{n,0} \hookrightarrow E^2_{n,0} = H_n(B),$$

(*)

[(*) is] called an edge homomorphism. (an alg. version exists for any first quadrant S.S. induced by a filtration).

[Similarly, no differentials can emanate from $E^r_{0,n}$, so each $E^{r+1}_{0,n} = E^r_{0,n} / \mathrm{im}(\partial_r : E^r_{r,n-r+1} \to E^r_{0,n})$

$E^r_{r,n-r+1} \equiv 0$ $r \geq n+2$ $\qquad$ $F_{-1} = 0$ b/c $B^{-1} = \emptyset$.

$$H_n(F) = E^2_{0,n} \twoheadrightarrow E^3_{0,n} \twoheadrightarrow \cdots \twoheadrightarrow E^{n+1}_{0,n} \twoheadrightarrow E^{n+2}_{0,n} = E^\infty_{0,n} = G_0 H_n(E) = F_0 H_n(E) \hookrightarrow H_n(E)$$

edge homomorphism]

Prop: The edge homomorphism of the L-S. S.S. $H_n(E) \to H_n(B)$ coincides with $\pi_* : H_n(E) \to H_n(B)$.

[Similarly, the edge homomorphism $H_n(F) \to H_n(E)$ coincides w/ i_*, where $i : F \hookrightarrow E$ incl. of a fiber].

Pf sketch: (Let's do the case of $H_n(E) \to H_n(B)$, other case exercise).

Consider first the trivial fibration $* \to B \xrightarrow[\pi = id]{} B$. Claim: the edge hom $H_n(B) \longrightarrow H_n(B)$ associated to the L-S. S.S. for this fibration is just $id_{H_n(B)}$. (exercise).

Next, for a general $F \to E \to B$ as above, there is a map of fibrations

$$\begin{tikzcd} F \arrow[r, "\text{project to point}"] \arrow[d] & * \arrow[d] \\ E \arrow[r, "\pi"] \arrow[d, "\pi"] & B \arrow[d, "id"] \\ B \arrow[r, "id"] & B \end{tikzcd}$$

which induces, by naturality of the L-S. S.S., a commutative diagram:

$$\begin{tikzcd} H_n(E) \arrow[r, "\pi_*"] \arrow[d, "\text{(edge hom. for } F\to E\to B)"'] & H_n(B) \arrow[d, "\text{edge hom. for } * \to B \to B) = id_{H_n(B)} \text{ by claim above.}"] \\ H_n(B) \arrow[r, "id"] & H_n(B). \end{tikzcd}$$

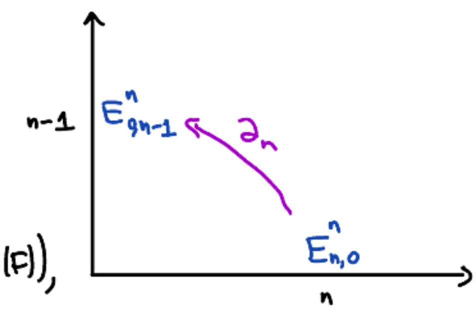
$\Rightarrow$ this edge homomorphism $\uparrow$ is just π_*. $\boxed{\Xi}$

Transgression: $F \to E \to B$ fibration.

Again, assume F, B path connected & $\{H_q(F_x)\}_{x \in B}$ trivial (e.g., if $\pi_1(B) = \{*\}$).

In the L-S. S.-S., the differentials

$$\partial_n : E^n_{n,0} \longrightarrow E^n_{0,n-1}$$



map the bottom horizontal edge (a subgroup of $H_n(B)$) to the left vertical edge (a quotient of $H_{n-1}(F)$), and are called *transgressions*.

$\partial_2 : E^2_{2,0} \longrightarrow E^2_{0,1}$, so in this case get a homomorphism from base to fiber. In general,

$$\underset{H_2(B)}{\overset{\shortparallel\wr}{E^2_{2,0}}} \qquad \underset{H_1(F)}{\overset{\shortparallel\wr}{E^2_{0,1}}}$$

$$H_n(B) = E^2_{n,0} \supseteq E^n_{n,0} \xrightarrow{\partial_n} E^n_{0,n-1} \twoheadleftarrow E^2_{0,n-1} = H_{n-1}(F).$$

An element $x \in E^2_{n,0} = H_n(B)$ that survives to $E^n_{n,0}$ (i.e., $\partial_2 x = 0, \partial_3 x = 0, \ldots \partial_{n-1} x = 0$) is called *transgressive*.

The basic claim is that the subgroup of transgressive elts. & the transgression map have a topological interpretation, as a fibration analogue of a connecting homomorphism.

How? Consider

$$\begin{tikzcd} & H_n(E,F) \arrow[r, "\partial_*"] \arrow[d, "\pi_*"] & H_{n-1}(F) \\ H_n(B) \arrow[r, "j_*"] & H_n(B,b) & \end{tikzcd}$$

fiber over basepoint b.

iso. for $n>0$.

($\pi:(E,F)\to(B,b)$ where $F:=$ fiber over b).

($\pi_*: H_n(E,F)/\ker\pi_* \xrightarrow{\cong} \mathrm{im}\,\pi_*$)

This induces a map from a subgroup of $H_n(B)$ to a quotient of $H_{n-1}(F)$ via:

(subgp of $H_n(B)$) (quotient of $H_{n-1}(F)$)

$$(\star)\quad \begin{tikzcd} H_n(B)\cong H_n(B,b)\supseteq \mathrm{im}(\pi_*) \arrow[r, "\cong", "(\pi_*)^{-1}"'] \arrow[rr, bend right, "\tau_*"'] & H_n(E,F)/\ker\pi_* \arrow[r, "\partial_*"] & \dfrac{H_{n-1}(F)}{\partial_*(\ker\pi_*)} & H_{n-1}(F). \arrow[l, two heads] \end{tikzcd}$$

Prop: The map τ_* coincides with the transgression $\partial_n: E^n_{n,0}\to E^n_{0,n-1}$, meaning:

$$\begin{tikzcd} E^n_{n,0} \arrow[r, phantom, "\cong"] \arrow[d, "\partial_n"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{im}(\pi_*) \arrow[d, "\tau_*"] \\ E^n_{0,n-1} \arrow[r, phantom, "\cong"] & H_{n-1}(F)/\partial_*(\ker\pi_*) \end{tikzcd}$$

(moreover $E^2_{n,0}=H_n(B)$, $\cup$ $\circlearrowleft$ $\cup$, $\underbrace{E^n_{n,0}=\mathrm{im}\,\pi_*}_{\star}$, $E^2_{0,n-1}=H_{n-1}(F)$, $\downarrow$ $\circlearrowleft$ $\downarrow$, $E^n_{0,n-1}=H_{n-1}(F)/\partial_*(\ker\pi_*)$)

We call τ_* the <u>transgression hom.</u> associated to the fibration; it coincides w/ transgression of the L.S.S.S. by Prop.

<u>Pf:</u> omitted (see [Hatcher-SS., Prop 1.13]), but to start eg, how to identify $\mathrm{im}(\pi_*)\cong E^n_{n,0}$ as in $\star$?

Consider map of fibration pairs: $\exists$ L.S.S.S. for a 'fibration pair' as below & naturality gives

$$\begin{array}{ccc} F & \to & (F,F) \\ \downarrow & & \downarrow \\ E & \to & (E,F) \\ \downarrow & & \downarrow \\ B & \to & (B,b) \end{array} \qquad E^*_{p,q}\to \bar{E}^*_{p,q}.$$

(Rmk: can set up L.S.S.S for fibration pairs in same way)

Now $H_0(B,b)=0$, so $\bar{E}^*_{0,q}\equiv 0$. $\Rightarrow \bar{E}^n_{n,0}=\bar{E}^\infty_{n,0}$ (as $\bar\partial_n:\bar{E}^n_{n,0}\to\bar{E}^n_{0,n-1}=0$ is 0).

(coincides w/ $\bar{E}^\infty_{n,0}\hookrightarrow\bar{E}^2_{n,0}$, which)

$\Rightarrow$ (by edge hom. prop): the image of the inclusion $\bar{E}^n_{n,0}\hookrightarrow\bar{E}^2_{n,0}$ is just $\mathrm{im}(\pi_*)\subseteq H_n(B,b)$

Now for $n>0$, $H_n(B,b)\xleftarrow{\cong}H_n(B)$, hence we learn

$$\begin{array}{ccc} \mathrm{im}(\pi_*) & \subset & H_n(B,b) \\ \| & & \| \\ \bar{E}^n_{n,0} & \hookrightarrow & \bar{E}^2_{n,0} \\ \cong & & \cong \\ E^n_{n,0} & \hookrightarrow & E^2_{n,0} \\ & & \| \\ & & H_n(B) \end{array}$$

, which implies $\star$.

$\blacksquare$

Using this, we can sketch a proof of the Hurewicz theorem (the mod $\mathcal{C}$ version is similar but requires slightly more elaborate arguments):

<u>Thm</u> (Hurewicz): Say X $(n-1)$-connected, i.e., $\pi_i(X)=\{0\}$ $i\le n-1$, $n\ge 2$. Then $\tilde{H}_i(X)=0$ $i\le n-1$, and the Hurewicz map $h:\pi_n(X)\to H_n(X)$ is an isomorphism. (Recall: $h:\pi_n(X)\to H_n(X)$, $[f:S^n\to X]\mapsto f_*[S^n]$).

Sketch: Fact from homotopy theory: any fibration $F \to E$, $E \downarrow B$ induces a LES in homotopy groups.

$\Rightarrow$ for $\Omega X \to PX$ (contractible), $PX \downarrow X$ have (LES in homotopy groups)

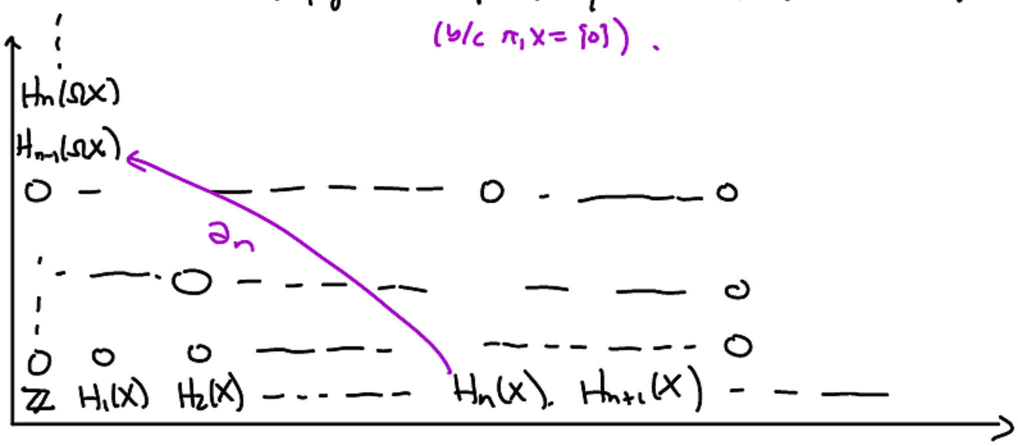
$$\cdots \to \underbrace{\pi_i(PX)}_{0} \to \pi_i(X) \xrightarrow{\partial_*} \pi_{i-1}(\Omega X) \to \underbrace{\pi_{i-1}(PX)}_{0}$$

$\Rightarrow \pi_i(X) \xrightarrow[\partial_*]{\cong} \pi_{i-1}(\Omega X)$. ΩX is therefore $(n-2)$-connected; we want to inductively use fact that Hurewicz holds for ΩX. (base case: $n=2 \Rightarrow$ use π_1 Hurewicz (Math 545) $\pi_1(\Omega X) \xrightarrow[\cong]{h} H_1(\Omega X)$, which holds as stated b/c $\pi_1(\Omega X) \cong \pi_2(X)$ is abelian).

(e.g., $\pi_0(\Omega_* X) \underset{\text{by def'n}}{=} \pi_1(X,*)$).

In homology, we have the L-S. S.S. (for $\Omega X \to PX \to X$) w/ page 2 $H_p(X; H_q(\Omega X))$; by induction $H_q(\Omega X) = 0$ $q \le n-2$ (as ΩX is $(n-2)$-connected). (b/c $\pi_1 X = \{0\}$).

$E^2_{p,q}$.



$\Rightarrow H_1(X) = H_2(X) = \cdots = H_{n-1}(X) = 0$ (as no ∂_i from it can be non-zero, & $E^\infty_{p,q} = 0$ for (p,q) non-zero),

In this case, $\partial_i = 0$ on $H_n(X)$ for each $i < n$, so all of $H_n(X)$ is transgressive, & the transgression ∂_n gives an iso. $H_n(X) \xrightarrow{\cong} H_{n-1}(\Omega X)$. (all survives to page n).

Using the above Prop. comparing ∂_n to τ_*, & def'n of $\partial_*: \pi_n(X) \to \pi_{n-1}(\Omega X)$ in LES, (omitted here) construct a commutative diagram (exercise) for such X:

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "\partial_*", "\cong"'] \arrow[d, "h"] & \pi_{n-1}(\Omega X) \arrow[d, "h"] \\
H_n(X) \arrow[r, "\tau_*", "\cong"'] & H_{n-1}(\Omega X)
\end{tikzcd}$$

iso, by induction (as ΩX $(n-2)$ connected); works even when $n=2$ b/c $\pi_1 \Omega X$ abelian. (usually π_1 Hurewicz only gives an iso. $\pi_1(X)^{ab} \xrightarrow{\cong} H_1(X)$).

$\Rightarrow h: \pi_n(X) \xrightarrow{\cong} H_n(X)$. ∎

Can use the L-S. S-S to reprove the Gysin exact sequence (in more generality, for arbitrary spherical fibrations), the Leray-Hirsch theorem, build other exact sequences, & prove many structural results.

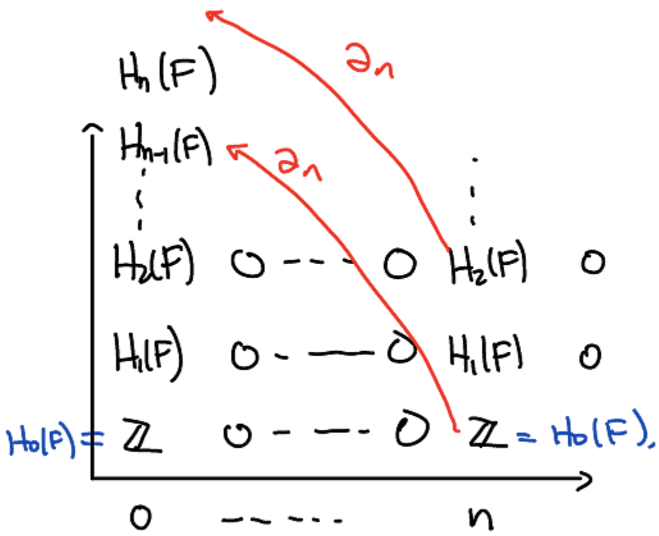
(e.g., using above interplay between L-S. S.S. & LES in homotopy groups, can learn a lot of information about homotopy groups too).

E.g.) <u>Wang sequence</u> Say have a fibration

$$\begin{array}{ccc} F & \to & E \\ & & \downarrow \\ & & S^n \end{array} \qquad n > 1 \ . \quad \text{(Say path-connected fiber).}$$

L-S.S-S has, as its E^2 page $\{H_p(S^n; H_q(F))\}_{(p,q)} = \begin{cases} H_q(F) & p = 0, n \\ 0 & \text{else.} \end{cases}$

$E^2_{p,q}$.



$\partial_i \equiv 0$ for $i < n$, so all terms survive to E^n. The only possible non-zero differential is

$$\partial_n : \underset{E^n_{n,i}}{\overset{\|}{H_i(F)}} \longrightarrow \underset{E^n_{0,i+n-1}.}{\overset{\|}{H_{i+n-1}(F)}}, \quad \text{i.e.,} \quad E^{n+1} = E^\infty = H^\bullet(\partial_n)..$$

So, there's a SES: $\boxed{0 \to E^\infty_{n,i} \xrightarrow{*} H_i(F) \xrightarrow{\partial_n} H_{i+n-1}(F) \xrightarrow{*} E^\infty_{0,i+n-1} \to 0.}$

In turn, $E^\infty_{n,i} = G_n H_{i+n}(E)$, $\quad E^\infty_{0,i+n-1} = G_0 H_{i+n-1}(E) = F_0 H_{i+n-1}(E)$.

$H_{i+n}(E)/F_0 H_{i+n}(E) = H_{i+n}(E)/F_{n-1} H_{i+n}(E) \twoheadleftarrow H_{i+n}(E)$ $\qquad\qquad \hookrightarrow H_{i+n-1}(E)$.

(note $S^n = e^0 \cup e^n$, so skeletal filtration is $F_0 C_*(E) = F_1 = F_2 = \cdots = F_{n-1} \subseteq F_n C_*(E) = C_*(E)$).

i.e., $\exists$ a SES $0 \to F_0 H_*(E) \to H_*(E) \to \left(H_{i+n}(E) / F_{n-1} H_{i+n}(E) \right) \to 0$

i.e., a SES $\boxed{0 \to G_0 H_*(E) \xrightarrow{(a)} H_*(E) \xrightarrow{(b)} G_n H_*(E) \to 0,}$ (**)

In particular, we can splice those SES's together to get:

$$\begin{tikzcd}[column sep=tiny] & & G_n H_{i+n}(E) \arrow[dr, "*"] & & & & G_0 H_{i+n-1}(E) \arrow[dr, "(a)"] & & G_n H_{i+n-1}(E) \arrow[dr, "*"] & & \\ \cdots \arrow[r] & H_{i+n}(E) \arrow[ur, "(b)"] \arrow[rr] & & H_i(F) \arrow[rr, "\partial_n"] & & H_{i+n-1}(F) \arrow[ur, "*"] \arrow[rr] & & H_{i+n-1}(E) \arrow[ur, "(b)"] \arrow[rr] & & H_{i-1}(F) \arrow[r, "\partial_n"] & \cdots \end{tikzcd}$$

(exactness at $H_*(E)$ follows from (**)).

This is called the <u>Wang LES</u>.