Last time's proof directly generalizes to (same exact proof)

Thm: (UCT/R): R any PID (e.g., $\mathbb{Z}$, any field), and $C_\bullet$ a chain complex of free R-modules, G another R-module. Then, $\exists$ SES

$$0 \to \mathrm{Ext}^{(1)}_{(R)}(H_{n-1}(C_\bullet), G) \to H^n(\mathrm{Hom}_R(C_\bullet, G)) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(C_\bullet), G) \to 0$$

natural in $C_\bullet$ and G, & split (not naturally split).

(R PID $\Rightarrow$ $0 \to B_n \to Z_n \to H_n \to 0$ gives a proj. resolution of H_n, for instance).

In particular, if we begin with $C_\bullet(X;R)$ $(:= C_\bullet(X) \otimes_{\mathbb{Z}} R)$, and

$$C^\bullet(X;R) := \mathrm{Hom}_{\mathbb{Z}}(C_\bullet(X), R) \cong \mathrm{Hom}_R(C_\bullet(X;R), R) \quad (\text{why?})$$

In particular, we can now compute $H^\bullet(X;R)$ in terms of $H_\bullet(X;R)$ using UCT/R.

Special case: $R = k$ a field (i.e., $\mathbb{Q}$, $\mathbb{Z}/2\mathbb{Z}$, etc.) then any k-module M is automatically free hence projective. $\Rightarrow \mathrm{Ext}^{(1)}_k(M,k) = 0$ (b/c $(0 \to M) \xrightarrow{\simeq} M$ is a proj. resolution)

$$\Rightarrow \boxed{H^n(X;k) \underset{\cong}{\xrightarrow{\beta}} \mathrm{Hom}_k(H_n(X;k), k) = H_n(X;k)^\vee} \quad \boxed{(\text{over a field}).}$$

Rmk: $R = \mathbb{Z}$. If H any abelian group, then $\mathrm{Ext}(H, \mathbb{Z}) \cong \text{Torsion part}(H)$. (follows from classification of abelian groups + computations last class).

$\Rightarrow$ there is a (non-canonical; using splitting) iso. $H^n(X;\mathbb{Z}) \cong \mathrm{Free}(H_n(X;\mathbb{Z})) \oplus \mathrm{Tors}(H_{n-1}(X;\mathbb{Z}))$

Example: $X = \mathbb{RP}^3$ recall that can compute $H_\bullet$ via cellular chains:

$$C_\bullet^{CW} = \{ \underset{\deg 0}{\mathbb{Z}} \xleftarrow{\times 0} \underset{\deg 1}{\mathbb{Z}} \xleftarrow{\times 2} \underset{\deg 2}{\mathbb{Z}} \xleftarrow{\times 0} \underset{\deg 3}{\mathbb{Z}} \}$$

$$\Rightarrow H_i(\mathbb{RP}^3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/2 & i=1 \\ 0 & i=2 \\ \mathbb{Z} & i=3 \\ 0 & \text{else.} \end{cases} \quad \underset{\text{UCT}}{\Rightarrow} \quad H^i(\mathbb{RP}^3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0,3 \\ \mathbb{Z}/2 & i=2 \\ 0 & \text{else.} \end{cases}$$

(note: by cohomological version of the $C_\bullet^{CW} \underset{\text{ch. htpy}}{\simeq} C_\bullet^{sing}$ argument, we can compute that $C^\bullet_{sing} \simeq C^\bullet_{CW}$

(exercise: verify why this is true).

equiv

$\mathrm{Hom}_{\mathbb{Z}}(C_\bullet^{cw}, \mathbb{Z})$)

In general, $H^\bullet$ has access to all of the LES tools that we had for homology, i.e.,

$(X,A) \rightsquigarrow$ LES for the pair (arrows reversed):

$$\cdots \to H^n(X,A) \to H^n(X) \to H^n(A) \xrightarrow{\delta} H^{n+1}(X,A) \to \cdots$$

this is induced by the SES $\quad 0 \to C^\bullet(X,A) \to C^\bullet(X) \to C^\bullet(A) \to 0$

which comes by dualizing $\quad 0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X,A) \to 0$

($\mathbb{Z}$-dual is a SES because $\underbrace{}$ is split).

similarly have Mayer-Vietoris, excision, ..

Rmk: Homology UCT involving Tor has a similar proof.

Note: • $\mathrm{Tor}^{(\mathbb{Z})}(\mathbb{Z}, G) := 0 \quad \big($use $\underbrace{(0 \to \mathbb{Z})}_{P_\bullet} \xrightarrow{\simeq} \mathbb{Z}\big)$

• $\mathrm{Tor}(\mathbb{Z}_m, G) \quad \big($use $\underbrace{(\mathbb{Z} \xrightarrow{\times m} \mathbb{Z})}_{P_\bullet} \xrightarrow{\simeq} \mathbb{Z}_m$

$$P_\bullet \otimes_{\mathbb{Z}} G := \underset{\deg 1}{G} \xrightarrow{\times m} \underset{\deg 0}{G}, \quad \text{so}$$

$$\mathrm{Tor}_{(0)} = \mathbb{Z}_m \otimes G = G/mG$$

$$\mathrm{Tor}_{(1)} = \ker(\times m) = \{m\text{-torsion subgroup of } G\}..$$

• exercise: $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n)$ & $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n)$.

Künneth theorems in homology and cohomology

Goal: understand relationship between $H_\bullet / H^\bullet$ of $X \times Y$ and $H_\bullet/H^\bullet$ of individual factors.

over a field, the result will state

tensor of graded abelian groups

• $H_\bullet(X\times Y; k) \cong H_\bullet(X;k) \otimes H_\bullet(Y;k)$

(means $H_n(X\times Y;k) \cong \bigoplus_{i+j=n} H_i(X;k) \otimes H_j(Y;k)$.)

• similar for cohomology, assuming at least one of X, Y finite type

(Z finite type if each $H_i(Z;k)$ is finitely generated). (e.g., CW cplx w/ finitely many cells in each dimension)

(basic problem is that $(V\otimes W)^\vee \not\cong V^\vee \otimes W^\vee$ in genl, it is if one of V, W is fin. dim'l).

• In genl over R, there's a map ($\leftarrow$) which fails to be an $\cong$ (coker is $\mathrm{Tor}(-,-)$)

Künneth is an immediate consequence of two results:

today $\rightarrow$ (1) The Eilenberg-Zilber theorem says $C_\bullet(X\times Y) \xrightarrow[\text{ch. htpy equiv}]{\simeq} C_\bullet(X)\otimes C_\bullet(Y)$

can take $\otimes$ of chain cplxes & get a chain cplx.

(2) The algebraic Künneth theorem comparing

$$H_\bullet(C_\bullet\otimes D_\bullet) \quad \text{to} \quad H_\bullet(C_\bullet)\otimes H_\bullet(D_\bullet)$$

(a Tor term appears).

generalizes Homology UCT in a way by allowing $D_\bullet$ to not just be $R_\bullet$

(reference: [Bredon])

Def: $C_\bullet$ and $D_\bullet$ chain complexes over R ($=\mathbb{Z}$ for now);

define $C_\bullet \otimes_{(R)} D_\bullet$ by $(C_\bullet\otimes D_\bullet)_n = \bigoplus_{i+j=n} C_i\otimes D_j$, with (tensor of graded abelian groups)

$$\partial_{C_\bullet\otimes D_\bullet}(\underset{\text{degree } i}{a}\otimes \underset{\text{degree } j}{b}) = \partial a\otimes b + (-1)^{\deg(a)=i}\, a\otimes \partial b .$$

can think of this as $\partial_{C_\bullet\otimes D_\bullet} = \partial\otimes \mathrm{id} + \mathrm{id}\otimes\partial$, using the convention

that $(f\otimes g)(a\otimes b) = (-1)^{\deg(g)\deg(a)} f(a)\otimes g(b)$

Recall, a chain homotopy equivalence between $A_\bullet$ and $B_\bullet$ consists of

$$A_\bullet \underset{g}{\overset{f}{\rightleftarrows}} B_\bullet \qquad f, g \text{ chain maps } \left(\text{e.g., } \begin{array}{l} f\circ\partial_A = \partial_B\circ f \\ g\circ\partial_B = \partial_A\circ g \end{array}\right)$$

with $f\circ g \underset{\text{ch. htpic}}{\simeq} \mathrm{id}_{B_\bullet}$ $\qquad g\circ f \underset{\text{ch. htpic}}{\simeq} \mathrm{id}_{C}$.

$\Rightarrow$ $[f]$, $[g]$ induce inverse isos. on $H_\bullet(A) \rightleftarrows H_\bullet(B)$.

Theorem: (Eilenberg-Zilber): There is a chain homotopy equivalence (over any coeffs. R)

Θ ← (particular choice often called) Alexander-Whitney map

$$C_\bullet(X\times Y) \underset{(-)\times(-)}{\overset{\Theta}{\rightleftarrows}} C_\bullet(X)\otimes C_\bullet(Y),$$

which is natural (functorial in X and Y), are unique up to chain homotopy.

"cross product"

cross product
(specific model often called) (Eilenberg – Zilber map).

To start, we need to define the maps. Let's begin with the cross product

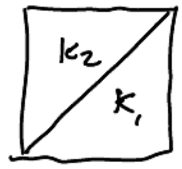
$$\times : C_p(X) \otimes C_q(Y) \longrightarrow C_{p+q}(X \times Y).$$

How to define?

Given a generator $\sigma : \Delta^p \to X$, $\tau : \Delta^q \to Y$, want "$\sigma \times \tau$" $\in C_{p+q}(X \times Y)$.

- Take the naive product $(\sigma, \tau) : \Delta^p \times \Delta^q \longrightarrow X \times Y$.

- if $p = 0$ or $q = 0$ then $\Delta^p \times \Delta^q \cong \Delta^{p+q}$ $\quad (\Delta^p \times \Delta^0 = \Delta^p)$.
  in this case, define $\sigma \times \tau := (\sigma, \tau)$



- In general, $\Delta^p \times \Delta^q$ is not a simplex, but it can be triangulated $\Delta^p \times \Delta^q = \bigcup k_i : \Delta^{p+q} \to \Delta^p \times \Delta^q$
  & roughly define $\sigma \times \tau := \sum (\sigma, \tau)|_{k_i}$.

$\Delta^1 \times \Delta^1$ can be triangulated.

Special case: 'prism operator' involves triangulating $\Delta^p \times \Delta^1$ for all p.
(used to show $f \simeq g \Rightarrow f_\# \underset{\text{ch. htpic}}{\simeq} g_\#$).

options:
has some advantages too (e.g., $\times$ is strictly associative)

- explicit formula (combinatorial, generalizes 'prism', get one $p+q$ simplex for each "shuffle" of $(v_0, \dots, v_p)$ & $(w_0, \dots, w_q)$ vertices of Δ^p & Δ^q)

we'll take this approach

- argue that such a map has to exist for general reasons, using "method of acyclic models" (proof technique used a lot in comparing homology theories: singular vs. simplicial vs. cubical etc.)

Thm (existence of $\times$): For each p, q, $\exists$ bilinear

$$\times : C_p(X) \times C_q(Y) \to C_{p+q}(X \times Y) \quad \text{such that:}$$

(1) For $x_0 : \Delta^0 \to X$, $x_0 \times \tau = (x_0, \tau) : \Delta^{0+q} = \Delta^q \to X \times Y$

Similarly, for $y_0 : \Delta^0 \to Y$, $\sigma \times y_0 = (\sigma, y_0)$.

(2) (naturality): If $f : X \to X'$ $g : Y \to Y'$ induces $(f,g) : X \times Y \to X' \times Y'$,
then $(f,g)_\# (a \times b) = (f_\# a) \times (f_\# b)$.

(3) (chain map/boundary formula): $\times$ is a chain map $C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$,

$$\partial(a\times b) = \partial a\times b + (-1)^{\deg(a)} a\times\partial b.$$

Pf: Induction on p, q.

- base case: have such maps when $p=0$ or $q=0$.

- Inductive step: fix $p>0$ and $q>0$ (so $p+q\geq 1$) & say $\times$ has been defined for all smaller $(p+q)$'s for all X and Y.

Want to define $\sigma\times\tau$ for $\sigma\in C_p(X)$, $\tau\in C_q(Y)$.

First define $\times$ on a very special p-simplex $\times$ a very special singular q-simplex in special spaces: namely consider

$$i_p : \Delta^p \xrightarrow{id} \Delta^p \qquad \rightsquigarrow \text{ give elements in } C_p(\Delta^p) \ \& \ C_q(\Delta^q) \text{ respectively.}$$
$$i_q : \Delta^q \xrightarrow{id} \Delta^q$$

Let's try to first define $i_p\times i_q \in C_{p+q}(\Delta^p\times\Delta^q)$. How?

By (3) we want $i_p\times i_q$ to satisfy:

call this expression α.

$$\underline{(*)} \quad \partial(\overbrace{\underbrace{i_p\times i_q}_{\text{not yet defined}}}^{\beta}) = \overbrace{\partial i_p \times i_q + (-1)^p i_p\times\partial i_q}^{\alpha}.$$

both inductively defined, as we've defined $\times$ on all $C_k(X)\otimes C_\ell(Y)$ for $k+\ell<p+q$.

Compute $\partial(RHS) = \partial(\alpha)$:

$$= \cancel{\partial\partial i_p\times i_q} + (-1)^{p-1}\partial i_p\times\partial i_q + (-1)^p\partial i_p\times\partial i_q + \cancel{i_p\times\partial\partial i_q} = 0$$

cancel.

So in fact α is a cycle in $C_{p+q-1}(\Delta^p\times\Delta^q)$.

We want $\alpha=\partial\beta$, i.e., want α to be a boundary.

"acyclic"

Since $p+q-1>0$ and $\Delta^p\times\Delta^q$ is contractible, $H_{p+q-1}(\underline{\Delta^p\times\Delta^q})=0$, so in fact $\exists$ a chain β with $\partial\beta=\alpha$.

Pick any such chain & call it $i_p\times i_q$.

What to do for a general $\sigma:\Delta^p\to X$, $\tau:\Delta^q\to Y$? In fact, $\sigma\times\tau$ is forced by naturality: note that as an element of $\sigma\in C_p(X)$, $\sigma = \sigma_\#\circ i_p$, $\sigma_\#: C_p(\Delta^p)\to C_p(X)$

$i_p \in C_p(\Delta^p)$.

$$\begin{tikzcd} \Delta^p \arrow[r, "i_p = id"] \arrow[rr, bend right, "\sigma"'] & \Delta^p \arrow[r, "\sigma"] & X \end{tikzcd}$$

Similarly, $\tau = \tau_{\#} \circ i_q$.

Hence if $(\sigma, \tau): \Delta^p \times \Delta^q \to X \times Y$ is the product map, by naturality (2), we get:

$$\underset{\text{(if defined)}}{\sigma \times \tau} = (\sigma_{\#} i_p) \times (\tau_{\#} i_q) \underset{\text{naturality}}{=\!=} (\sigma, \tau)_{\#} (\underline{i_p \times i_q})$$

defined above!

In order for naturality to hold, we must define $\sigma \times \tau := (\sigma, \tau)_{\#} (i_p \times i_q)$.

($i_p : \underline{\Delta^p \to \Delta^p}$, $i_q : \underline{\Delta^q \to \Delta^q}$ are the "models").

Check that this definition satisfies the boundary formula:

compute $\partial(\sigma \times \tau) = \partial((\sigma, \tau)_{\#} (i_p \times i_q)) = (\sigma, \tau)_{\#} (\partial (i_p \times i_q))$

$$\underset{\text{boundary formula for } i_p \times i_q}{=\!=} (\cdots) \underset{\text{(exercise).}}{=} \cdots = \partial\sigma \times \tau + (-1)^{\deg(\sigma)} \sigma \times \partial\tau.$$

$\boxtimes$

<u>Rmk</u>: Can use method acyclic models to inexplicitly prove barycentric subdivision too.

<u>Note</u>: For pairs, define $(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$.

check/note that $\times$ naturally takes $C_\bullet(X, A) \otimes C_\bullet(Y, B)$ into $C_\bullet((X, A) \times (Y, B))$.

(e.g., $A \subset X$, then $\times : C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$ carries $C_\bullet(A) \otimes C_\bullet(Y)$ to $C_\bullet(A \times Y)$).

The map θ (the other way):

<u>Technical lemma</u>: Say X, Y are contractible, then $C_\bullet(X) \otimes C_\bullet(Y)$ is acyclic

i.e., $H_n(-) \begin{cases} = 0 & \text{for } n > 0 \\ = \mathbb{Z} & \text{for } n = 0, \text{ generated by } [x_0 \otimes y_0], \end{cases}$

$x_0 : \Delta^0 \to X$, $y_0 : \Delta^0 \to Y$ any points.

<u>Pf sketch</u>: X contractible $\iff$ (*) $\underset{pr}{\overset{x_0}{\rightleftarrows}} X$ are homotopy inverse, in particular

$X \xrightarrow{pr} \{*\} \xrightarrow{x_0} X$ ...

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "\varepsilon_X"'] & \{*\} \arrow[r] & X \end{tikzcd}$$ ε_X is homotopic to id_X.

similarly $\varepsilon_Y : Y \xrightarrow{pr} \{*\} \xrightarrow{y_0} Y$ is htpic to id_Y.

$\Rightarrow$ $\exists$ chain homotopies H_X (on $C_\bullet(X)$) between $(\varepsilon_X)_\#$ and $\mathrm{id}_{C_\bullet(X)} = (\mathrm{id}_X)_\#$ (deg +1)

H_Y (on $C_\bullet(Y)$) between $(\varepsilon_Y)_\#$ and $\mathrm{id}_{C_\bullet(Y)} = (\mathrm{id}_Y)_\#$. (deg +1)

i.e., $\partial H_X + H_X \partial = \mathrm{id} - (\varepsilon_X)_\#$, same for H_Y, ε_Y.

$\star$ ($\overset{\text{e.g.,}}{\Rightarrow}$ $[\mathrm{id}] = (\varepsilon_X)_* : H_\bullet(X) \to H_\bullet(*) \to H_\bullet(X)$ $\Rightarrow$ $H_\bullet(X) = 0$ in deg > 0
X connected so $H_0(X) = \mathbb{Z}$)

same for Y,

Let $H_\otimes := H_X \otimes \mathrm{id}_{C_\bullet(Y)} + (\varepsilon_X)_\# \otimes H_Y$ on $C_\bullet(X) \otimes C_\bullet(Y)$ (deg +1 map)

"contract X to a point"   already contracted id_X to ε_X, now contract Y down.

Exercise: • $\partial_{C_\bullet(X)\otimes C_\bullet(Y)} H_\otimes + H_\otimes \circ \partial_{C_\bullet(X)\otimes C_\bullet(Y)} = \mathrm{id}\otimes\mathrm{id} - \varepsilon_X \otimes \varepsilon_Y$.

• finish the proof from here, using analogue of $\star$. ☒.

Next time: we'll sketch construction of Θ, & finish proof of Eilenberg-Zilber, sketch alg. Künneth.

1/27/2021

Thm: (existence of Θ): $\exists$ $\Theta : C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$ satisfying:

(1) Θ is a chain map.

(2) Θ is natural in X & Y

(i.e, $f : X \to X'$, $g : Y \to Y'$ then $(f,g) : X \times Y \to X' \times Y'$ and $\underline{\Theta \circ (f,g)_\# = (f_\# \otimes g_\#) \circ \Theta}$.).

(3) In degree 0, Θ is the following (determined) map:

$$\{(x,y) : \Delta^0 \cong \Delta^0 \times \Delta^0 \to X \times Y\} \longmapsto (x : \Delta^0 \to X) \otimes (y : \Delta^0 \to Y)$$

$$\underset{C_0(X \times Y)}{\cap} \qquad\qquad \underset{C_0(X) \otimes C_0(Y)}{\cap}.$$

Pf: again induction, appealing to the method of acyclic models.

• base case: (deg 0): defined by (3).

• say Θ defined in degrees $< k$, is a (natural) chain map, for all X, Y. To define Θ in degree k, first consider the special case

$$X = \Delta^k = Y, \text{ with special singular simplex } d_k : \Delta^k \xrightarrow{(id, id)} \overbrace{\Delta^k}^{X} \times \overbrace{\Delta^k}^{Y}$$ diagonal singular simplex.

so $d_k \in C_k(\Delta^k \times \Delta^k)$.

By induction, we've defined

$$\Theta(\partial d_k) \in \underbrace{\left(C_\cdot(\Delta^k) \otimes C_\cdot(\Delta^k)\right)_{k-1}}_{},$$ and we can check directly that

claim: $\Theta(\partial d_k)$ is a cycle in $\uparrow$. Follows from $\partial_\otimes \Theta(\partial d_k) \underset{\substack{\Theta \text{ chain} \\ \text{map} \\ \text{in degree } k-1}}{=} \Theta(\partial \cancel{\partial} d_k) = 0$.

We are seeking to define a $\Theta(d_k)$ chain satisfying chain eq'n,

$$\partial(\underbrace{\Theta(d_k)}_{\substack{\uparrow \\ \text{not yet defined;}}}) = \underbrace{\Theta(\partial d_k)}_{\text{defined inductively, \& is a cycle by above}}$$

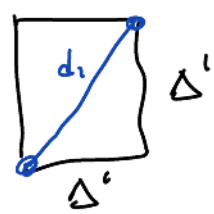
but if $[\Theta(\partial d_k)] = 0$ in $H_{k-1}(C_\cdot(\Delta^k) \otimes C_\cdot(\Delta^k))$, can pick any chain β w/ $\boxed{\partial \beta = \Theta(\partial d_k)}$ $\star$ & set $\Theta(d_k) = \beta$. (choice).

If $k > 1$, technical lemma $\Rightarrow H_{k-1}(C_\cdot(\Delta^k) \otimes C_\cdot(\Delta^k)) = 0$ b/c Δ^k, Δ^k acyclic.

$\Rightarrow$ such a β exists.

If $k = 1$, $H_0(C_\cdot(\Delta^k) \otimes C_\cdot(\Delta^k)) = \mathbb{Z}$, but we can directly compute that $[\Theta(\partial d_1)] = 0$, therefore a β exists.

$\Delta^1 \times \Delta^1$



$$\left(\begin{array}{l} \| \\ [\Theta((x_1, y_1) - (x_0, y_0))] \\ \| \, (3) \\ [x_1 \otimes y_1 - x_0 \otimes y_0] \\ = 0 \text{ by technical lemma.} \end{array}\right)$$

$\Theta(d_k) :=$ any choice of such β satisfying $\star$.

General X, Y, $\sigma : \Delta^k \longrightarrow X \times Y$ singular simplex:

notation: $\pi_X : X \times Y \to X$ projection, resp. $\pi_Y : X \times Y \longrightarrow Y$.

Note that $\pi_X \sigma : \Delta^k \longrightarrow X$, $\pi_Y \sigma : \Delta^k \longrightarrow Y$ gives

$$(\pi_X \sigma, \pi_Y \sigma) : \Delta^k \times \Delta^k \longrightarrow X \times Y,$$ w/ σ factoring as

$$\Delta^k \xrightarrow[(id, id)]{d_k} \Delta^k \times \Delta^k \xrightarrow{(\pi_X \sigma, \pi_Y \sigma)} X \times Y.$$

So $\sigma = (\pi_X \sigma, \pi_Y \sigma)_\# d_k$.

Hence by naturality, $\Theta(\sigma)$ should satisfy:

$$\boxed{\Theta(\sigma) = \Theta((\pi_X\sigma, \pi_Y\sigma)_\# d_k) = \underbrace{((\pi_X\sigma)\otimes(\pi_Y\sigma))}_{\text{defined.}}\underbrace{(\Theta(d_k))}_{\text{already defined.}}}$$

Hence, we can simply use $\uparrow$ to define $\Theta(\sigma)$.

Exercise: check this defn satisfies (1)→(3) in particular (1) & (2). ⊠.

We've defined $\Theta : C_\cdot(X\times Y) \to C_\cdot(X)\otimes C_\cdot(Y)$ and $\times : C_\cdot(X)\otimes C_\cdot(Y) \to C_\cdot(X\times Y)$

Q: are they homotopy inverses? what if we made different 'choices' of Θ, $\times$ (by choosing different bounding chains?)

Thm: Any two natural chain maps, either

- from $C_\cdot(X\times Y)$ to itself (e.g., $id_{C_\cdot(X\times Y)}$, $(-\times-)\circ\Theta$)
- from $C_\cdot(X)\otimes C_\cdot(Y)$ to itself (e.g., $\Theta\circ(-\times-)$, $id_{C_\cdot(X)\otimes C_\cdot(Y)}$)
- from $C_\cdot(X\times Y)$ to $C_\cdot(X)\otimes C_\cdot(Y)$, (e.g., Θ, another choice Θ')
- from $C_\cdot(X)\otimes C_\cdot(Y)$ to $C_\cdot(X\times Y)$ ($\times$, another choice of $\times'$).

that coincide w/ the canonical fixed maps in degree 0, are chain homotopic.

in deg 0:
$id_0 : C_0(X\times Y) \to C_0(X\times Y)$
$id_0 : C_0(X)\otimes C_0(Y) \to C_0(X)\otimes C_0(Y)$
$(x,y) : \Delta^0 \to X\times Y \longmapsto \{x\}\otimes\{y\}$
$\{x\}\otimes\{y\} \longmapsto (x,y)$.

Cor: Eilenberg-Zilber theorem as stated: $\exists$ natural chain htpy equiv. $C_\cdot(X\times Y) \underset{\times}{\overset{\Theta}{\rightleftarrows}} C_\cdot(X)\otimes C_\cdot(Y)$, w/ Θ, $\times$ unique up to chain homotopy.

Pf sketch of theorem: All 4 cases are similar, & all use method of acyclic models. Use the "models"
- $i_p \otimes i_q \in C_p(\Delta^p)\otimes C_q(\Delta^q)$ when starting from $C_\cdot(X)\otimes C_\cdot(Y)$
- $d_p \in C_p(\Delta^p\times\Delta^p)$ when starting from $C_\cdot(X\times Y)$.

In each case, given a pair ϕ, ψ of natural maps, coinciding in degree 0, try to construct a chain htpy D inductively satisfying $\partial D + D\partial = \phi - \psi$. Again inductively in each degree first construct D(model chain), then push forward. D(model chain) should satisfy

$$\partial D(\text{model chain}) = \phi(\text{model chain}) - \psi(\text{model chain}) - \underbrace{D\partial(\text{model chain})}_{\text{inductively already constructed.}}$$

As long as we know RHS is a cycle, & relevant $H_\cdot$(model space) is either 0 or at least $[RHS] = 0$ in $H_\cdot$, then a chain β satisfying $\partial\beta = RHS$ exists, & pick such a β & call it D(model chain).

Now 'push forward' to define D(any chain) b/c every chain is pushed forward from model.

Exercise: use this to spell out the details in 1-2 cases above. $\boxed{\text{E}}$.

Eilenberg-Zilber implies

recall coeffs. allowed in arguments above

$$H_p(X\times Y;R) \cong H_p(C_\bullet(X\times Y;R)) \underset{(EZ)}{\cong} \underline{H_p(C_\bullet(X;R)\otimes_R C_\bullet(Y;R))}$$

|| (if one desires)

$$H_p(C_\bullet(X)\otimes_{\mathbb{Z}} C_\bullet(Y)\otimes_{\mathbb{Z}} R).$$

Q1: can we analyze RHS in terms of $\otimes$ of Homology groups?

By dualizing on chain level, one gets ch. htpy equivalences:

R-coeffs.

$$\mathrm{Hom}_R(C_\bullet(X\times Y;R),R) \rightleftarrows \mathrm{Hom}_R(C_\bullet(X)\otimes C_\bullet(Y),R)$$

$$||$$

$$C^\bullet(X\times Y;R)$$

$$\rightsquigarrow H^\bullet(X\times Y;R) \cong H^\bullet(\underline{\mathrm{Hom}_R(C_\bullet(X)\otimes C_\bullet(Y),R)})$$

Q2: how does this compare to $\otimes$ of cohomologies?

Regarding Q1, we have

generalizes homology UCT

e.g., $\mathbb{Z}$ or a field.

Thm: (Algebraic Künneth theorem): let $K_\bullet, L_\bullet$ free chain cplxes (over any PID R), then $\exists$ a natural in $K_\bullet, L_\bullet$ SES; for each n

$$0 \to (H_\bullet(K_\bullet)\otimes H_\bullet(L_\bullet))_n \xrightarrow{\alpha} H_n(K_\bullet\otimes L_\bullet) \to \mathrm{Tor}_{(1)}^{(R)}(H_\bullet(K_\bullet),H_\bullet(L_\bullet))_{n-1}\to 0,$$

means $\bigoplus_{i+j=n} H_i(K_\bullet)\otimes H_j(L_\bullet)$

the standard map $[a]\otimes[b]\mapsto[a\otimes b]$.

means $\bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(K_\bullet),H_j(L_\bullet))$.

& it splits (non-naturally).

Pf has the same idea as proof of cohomology UCT; study failure of α to be injective via analyzing elements of $\mathrm{coker}(\alpha)$ (instead of "$\ker(\beta)$"). (omitted).

Cor: (of E-Z + Alg Künneth): Künneth theorem for homology: R PID, implicitly take R-coefficients. Then there is a natural SES (which splits, but non-naturally):

$$0 \to \bigoplus_{p+q=n} H_p(X)\otimes H_q(Y) \xrightarrow{[\times]} H_n(X\times Y) \to \bigoplus_{i+j=n-1}\mathrm{Tor}_1^R(H_i(X),H_j(Y)) \to 0$$

$$H_n(X\times Y) \cong H_n(C_\bullet(X\times Y)) \xleftarrow[\cong\, EZ]{\times} H_n(C_\bullet(X)\otimes C_\bullet(Y))$$

map appearing in alg Künneth.

If $R=k$ is a field, we know all Tor_1^k's are 0.

$\Rightarrow$ Künneth isomorphism: $\boxed{[\times]: H_\bullet(X,k)\otimes H_\bullet(Y,k) \xrightarrow{\cong} H_\bullet(X\times Y;k).}$

Example: Compute $H_\bullet(\mathbb{R}P^3\times \mathbb{R}P^3,k)$, for k any field.

Künneth: $\xrightarrow{\cong} H_\bullet(\mathbb{R}P^3,k)\otimes H_\bullet(\mathbb{R}P^3,k)$.

we know $\mathbb{R}P^3$ has CW homology chain complex (w/ R-coeffs):

$$\begin{array}{ccccccc} \deg 0 & & \deg 1 & & \deg 2 & & \deg 3 \\ R & \xleftarrow{\times 0} & R & \xleftarrow{\times 2} & R & \xleftarrow{\times 0} & R \end{array}$$

$$\Rightarrow H_i(\mathbb{R}P^3,R) = \begin{cases} R & i=0,3 \\ R/2R & i=1 \\ 2\text{-torsion}(R) & i=2 \\ 0 & \text{else} \end{cases} \underset{k}{\overset{\text{over a field}}{=}} \begin{cases} \mathrm{char}(k)=2 & \begin{cases} k & i=0,1,2,3 \\ 0 & \text{else.}\end{cases} \\ \mathrm{char}(k)\neq 2 & \begin{cases} k & i=0,3 \\ 0 & \text{else.}\end{cases} \end{cases}$$

over $\mathbb{Z}$: $\begin{pmatrix} \mathbb{Z} & i=0,3 \\ \mathbb{Z}/2 & i=1 \\ 0 & i=2. \end{pmatrix}$

in this case $\cong H_\bullet(S^3,k)$ $\leftarrow$ char $k\neq 2$.

get, for $\mathrm{char}(k)=2$:

$H_p(\mathbb{R}P^3)\otimes H_q(\mathbb{R}P^3)$:

p \ q	0	1	2	3
0	k	k	k	k
1	k	k	k	k
2	k	k	k	k
3	k	k	k	k

deg 0, deg 1, deg 2, deg 3, deg 4, deg 5, deg 6

$H_\bullet(\mathbb{R}P^3\times\mathbb{R}P^3)$ is:

deg	
0	k
1	$k\oplus k$
2	$k\oplus k\oplus k$
3	$k\oplus k\oplus k\oplus k$
4	$k\oplus k\oplus k$
5	$k\oplus k$
6	k

$\mathrm{char}(k)\neq 2$:

	0	1	2	3
0	k	0	0	k
1	0	0	0	0
2	0	0	0	0
3	k	0	0	k

$$\Rightarrow H_\bullet(\mathbb{R}P^3\times\mathbb{R}P^3,k) = \begin{cases} k & i=0,6 \\ k\oplus k & i=3 \\ 0 & \text{else.}\end{cases}$$

Exercise: compute $H_i(\mathbb{R}P^3\times\mathbb{R}P^3,\mathbb{Z})$. (using splitting of SES) There's a Tor term appearing, & in sum one gets:

$H_i \otimes_{\mathbb{Z}} H_j$

	0	1	2	3
0	$\cancel{\mathbb{Z}}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
2	0	0	0	0
3	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\cancel{\mathbb{Z}}$

only one Tor, $\operatorname{Tor}_2^{\mathbb{Z}}(\underset{H_1}{\mathbb{Z}/2}, \underset{H_1}{\mathbb{Z}/2}) \cong \mathbb{Z}/2$, contributes to $H_3(\mathbb{RP}^3 \times \mathbb{RP}^3; \mathbb{Z})$.

$\Rightarrow$ get:

i	$H_i(\mathbb{RP}^3 \times \mathbb{RP}^3; \mathbb{Z})$
0	$\mathbb{Z}$
1	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
2	$\mathbb{Z}/2$
3	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$
4	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
5	0
6	$\mathbb{Z}$

Künneth for cohomology (start):

implicitly R-coeffs.

Observe that the map $\theta : C_\bullet(X \times Y) \longrightarrow C_\bullet(X) \otimes C_\bullet(Y)$ (from E-Z theorem) induces, by dualizing, a map

$$\operatorname{Hom}_R(C_\bullet(X) \otimes C_\bullet(Y), R) \xrightarrow{\theta^*} \operatorname{Hom}_R(C_\bullet(X \times Y), R) = C^\bullet(X \times Y; R).$$
$$\Phi \longmapsto \Phi \circ \theta.$$

this is not necessarily equal to $C^\bullet(X;R) \otimes C^\bullet(Y;R) = \operatorname{Hom}_R(C_\bullet(X), R) \otimes \operatorname{Hom}_R(C_\bullet(Y), R)$.

Using the fact that R is a ring, can define for any two R-modules M, N a map

$$\operatorname{Hom}_R(M, R) \otimes \operatorname{Hom}_R(N, R) \longrightarrow \operatorname{Hom}(M \otimes N, R),$$
$$(f, g) \longmapsto \underbrace{\{ m \otimes n \longmapsto \underset{R \otimes R \xrightarrow[\text{mult.}]{\cong} R \quad m: R \times R \to R}{f(m) \otimes g(n)} \longmapsto f(m) \cdot g(n) \}}_{\text{we'll call this } m \circ (f \otimes g), \text{ or just } \underline{f \otimes g}}.$$

Using this, we get a map

$$\operatorname{Hom}_R(C_\bullet(X), R) \otimes \operatorname{Hom}_R(C_\bullet(Y), R) \longrightarrow \operatorname{Hom}_R(C_\bullet(X) \otimes C_\bullet(Y), R) \xrightarrow{\theta^*} \operatorname{Hom}_R(C_\bullet(X \times Y), R)$$
$$(f, g) \longmapsto f \otimes g \longmapsto (f \otimes g) \circ \theta$$

$$\begin{array}{ccc} \| & & \| \\ C^\bullet(X;R)\otimes C^\bullet(Y;R) & \longrightarrow & C^\bullet(X\times Y;R) \end{array}$$

Def: call this map the <u>cohomology cross product</u>, $\times$.

<u>Lemma</u> (omitted): $\delta(f\times g) = \underline{\delta f\times g + (-1)^{\deg(f)} f\times \delta g}$.

(Rmk: for the above to be true w/ signs, use a different convention $\delta f = (-1)^{\deg(f)+1} f\circ\partial$).

(rather than $\delta f = f\circ\partial$)

Also, $\times$ is <u>natural</u> with respect to maps $f: X\to X'$, $g: Y\to Y'$ (exercise: spell out)

& canonical (ind. of choice of Θ) — follows from analogous statement for Θ:

up to chain homotopy — up to chain homotopy!

(special case of)

Using this, next we articulate Künneth thm in cohomology. (requiring more finiteness hypotheses):

b/c

$$\underline{\mathrm{Hom}(V,k)}\otimes \underline{\mathrm{Hom}(W,k)} \longrightarrow \mathrm{Hom}(V\otimes W,k)$$
$$(f,g)\longmapsto f\otimes g$$

is <u>not</u> an iso. unless one of V,W ~~...~~ is finite-dimensional.