Rmk: there's a canonical element $1 \in C^0(X;R)$ defined by $1(x:\Delta^0 \to X) := 1 \in R$, (i.e., constant function).

This can be thought of as pulled back from $1 \in C^0(\{pt\};R)$ $(1(\{pt.\}) = 1)$, via $X \xrightarrow{\varepsilon} pt$, i.e., $\varepsilon^* 1_{pt} = 1_X$.

Claim: Given X, Y spaces, recall have $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$, and $- \times - : C^\bullet(X) \otimes C^\bullet(Y) \to C^\bullet(X \times Y)$, then

$$\underbrace{\alpha \times 1_Y = \pi_X^* \alpha,}\quad \text{and similarly} \quad 1_X \times \beta = \pi_Y^* \beta.$$

$\{\varepsilon : Y \to pt.$

To see e.g., $\curvearrowright$, we'll make use of the fact that $1_Y = \varepsilon^* 1_{pt}$, so by naturality

$$\alpha \times 1_Y = \alpha \times \varepsilon^* 1_{pt} \underset{\text{naturality}}{=\!=} (id_X, \varepsilon)^* (\alpha \times 1_{pt})$$

$$\begin{tikzcd} X \times Y \arrow[r, "{(id_X, \varepsilon)}"] \arrow[dr, "\pi_X"'] & X \times pt. \arrow[d, phantom, "\cong"] \\ & X \end{tikzcd}$$

(the identification $X \times pt \cong X$ sends $\alpha \times 1_{pt.} \longleftrightarrow \alpha$).

$$= \pi_X^*(\alpha).$$

X, Y spaces.

Thm (künneth for cohomology): If $R = \Lambda$ is a field and $\overbrace{H_i(X_{\Lambda}) \text{ is finite-rank for each } i}^{\text{"X is finite type over } \Lambda\text{"}}$ (or Y can be instead of X), then the cross product induces an isomorphism

$$- \times - : H^\bullet(X;\Lambda) \otimes H^\bullet(Y;\Lambda) \xrightarrow{\cong} H^\bullet(X \times Y;\Lambda).$$

(this also holds over a ring R as stated provided each $H_i(X)$ is a finitely generated, free R-module)

Sketch of proof: Over a field, UCT simplifies & gives:

$$\begin{tikzcd} H^\bullet(\mathrm{Hom}_\Lambda(C_\bullet(X) \otimes C_\bullet(Y), \Lambda)) \arrow[r, "\cong", "\text{UCT (coh.)}"'] & \mathrm{Hom}_\Lambda(H_\bullet(\underbrace{C_\bullet(X) \otimes C(Y)}), \Lambda) \arrow[d, "\text{alg. künneth (also simplifies)}"] \\ & \mathrm{Hom}(H_\bullet(X) \otimes H_\bullet(Y), \Lambda). \end{tikzcd}$$

$C_\bullet(X \times Y) \cong \text{EZ}$

$\cong \mathrm{Hom}(H_\bullet(X \times Y), \Lambda) \overset{\cong}{\uparrow} H^\bullet(X \times Y, \Lambda).$

Also, simplified alg. künneth implies

$$H^\bullet(C^\bullet(X;\Lambda) \otimes C^\bullet(Y;\Lambda)) \xleftarrow{\cong} H^\bullet(X;\Lambda) \otimes H^\bullet(Y;\Lambda).$$

All together by using these isomorphisms, we can show the cohomological cross product factors as:

$$\begin{tikzcd}
H^\cdot(X;\Lambda) \otimes H^\cdot(Y;\Lambda) \arrow[rr, "\times"] \arrow[d, "\cong"] & & H^\bullet(X\times Y;\Lambda) \arrow[d, "\cong"'] \arrow[d, phantom, "\text{uses: simplified UCT}", shift left=8ex] \\
\mathrm{Hom}_\Lambda(\underline{H_\cdot(X)},\Lambda) \otimes \mathrm{Hom}_\Lambda(\underline{H_\cdot(Y)},\Lambda) \arrow[dr, "\otimes", "(\mathrm{mult.})"'] & & \mathrm{Hom}(\underline{H_\cdot(X\times Y),\Lambda}) \\
& \mathrm{Hom}(H_\cdot(X)\otimes H_\cdot(Y),\Lambda) \arrow[ur, "\cong", "{[\theta]^*}"'] &
\end{tikzcd}$$

uses simplified homology-Künneth.

Therefore $\times$ is an isomorphism $\underline{\text{iff}}$ $\otimes$ is.

$\underline{\text{Obs}}$: $\otimes : \mathrm{Hom}_\Lambda(V,\Lambda) \otimes \mathrm{Hom}_\Lambda(W,\Lambda) \to \mathrm{Hom}(V\otimes W,\Lambda)$ is an iso. if one of V, W are finite rank in each degree.

graded vector spaces

$\underline{\text{(exercise)}}$.

$\underline{\text{How can it fail?}}$ Consider $V = \bigoplus_{i\in\mathbb{Z}} \Lambda = W$ $(= \Lambda^\infty)$ (direct $\underline{\text{sum}}$).

(arise topologically via $X = Y = \coprod_{i\in\mathbb{Z}} \mathrm{pt.}$)

Then, $\mathrm{Hom}_\Lambda(V\otimes W, \Lambda) \cong \mathrm{Hom}_\Lambda(V, \mathrm{Hom}_\Lambda(W,\Lambda)) = \mathrm{Hom}(V, W^*)$

$= \mathrm{Hom}(\Lambda^\infty, \prod_{i\in\mathbb{Z}} \Lambda)$.

(Q, R vec. spaces $\exists$ canonical map $Q^*\otimes R \xrightarrow{ev} \mathrm{Hom}(Q,R)$, sending $Q^*\otimes R$ to $\mathrm{Hom}_{\text{finite rank}}(Q,R)$)

But $\mathrm{Hom}_\Lambda(V,\Lambda)\otimes\mathrm{Hom}_\Lambda(W,\Lambda) = V^*\otimes W^* \xrightarrow[ev]{} \mathrm{Hom}(V,W^*)$,

$V = Q \quad W^* = R$

(this is the map $\otimes$ above, under identification $\mathrm{Hom}(V,W^*) \cong \mathrm{Hom}(V\otimes W,\Lambda)$).

compares $V^*\otimes W^*$ isomorphically to

$\mathrm{Hom}_{\text{finite rank}}(V, W^*)$. $\neq$ $\mathrm{Hom}_{(\text{all})}(V,W^*)$.

linear maps which have finite-dimensional image.

$\underline{\text{The cup product on cohomology}}$

Recall any top. space X has a diagonal map $\Delta : X \longrightarrow X\times X$
$x \longmapsto (x,x)$.

On homology, we could use this to get a "coproduct":

$$C_\cdot(X) \xrightarrow{\Delta_\#} C_\cdot(X\times X) \xrightarrow[\theta]{\simeq} C_\cdot(X)\otimes C_\cdot(X),$$ which essentially dualizes to:

$\underline{\text{Def}}$: The $\underline{\text{cup product}}$ on singular co-chains (w/ arbitrary coeffs. in some R), denoted $\smile$, is defined as;

$$\begin{tikzcd}
C^\bullet(X;R) \otimes C^\bullet(X;R) \arrow[rr, "\times"] \arrow[dr, dashed, "\text{mult.}\ (\otimes)"'] & & C^\bullet(X\times X;R) \arrow[r, "\Delta^\#"] & C^\bullet(X;R). \\
& \mathrm{Hom}(C_\bullet(X)\otimes C_\bullet(X);R) \arrow[ur, dashed, "\Theta^*"'] & &
\end{tikzcd}$$

cup product "$\alpha \cup \beta$"

Since $-\times-$ and $\Delta^\#$ are (co)-chain maps, $\cup$ is too, hence it induces a cohomology-level map, also call $\cup$ (by abuse of notation)

Thm: (properties of the cup product on <u>cohomology</u>)

(1) $\cup$ is natural, meaning if $f: X \to Y$, then $f^*(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta)$.

(<u>exercise</u>: <u>immediate</u> (b/c $\times$ is natural, and $f: X\to Y$ induces
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\
X\times X \arrow[r, "(f{,}f)"'] & Y\times Y
\end{tikzcd}$$
))

(2) $\alpha \cup 1 = \alpha = 1 \cup \alpha$ for any α. (the element 1 is a <u>unit</u> for $\cup$).

(3) $\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$. (associativity)

these all follow on <u>chain-level</u> from a particular chain model of $\cup$ (though both can be proved without this)

(4) $\alpha \cup \beta = (-1)^{\deg(\beta)\deg(\alpha)} \beta \cup \alpha$. ← on cohomology! (graded commutativity)

(typically cannot realize commutativity on chain level over arbitrary R).

(5) If (X,A) pair, and $i: A \hookrightarrow X$, so $i^*: H^\bullet(X) \to H^\bullet(A)$,

$\delta: H^p(A) \to H^{p+1}(X,A)$ connecting map, then

$$\delta(\underbrace{\alpha \cup i^*(\beta)}_{\overset{\cap}{H^\bullet(A)}}) = \underbrace{\delta(\alpha) \cup \beta}_{H^\bullet(X,A)}.$$

← uses the fact that $\cup$ defines a product on relative co-chains
$$C^\bullet(X,A) \times C^\bullet(X,A) \longrightarrow C^\bullet(X,A).$$

why? $C^\bullet(X,A) := \mathrm{Hom}_R(C_\bullet(X,A),R) = \mathrm{Hom}_R\left(\frac{C_\bullet(X)}{C_\bullet(A)}, R\right)$

$\cong$

$\mathrm{Hom}_R(C_\bullet(X),R)_{\mathrm{Ann}(C_\bullet(A))}$

look at $\phi: C_\bullet(X) \to R$ w/ $\phi|_{C_\bullet(A)} \equiv 0$.

so, need to check if α, β annihilate $C_\bullet(A)$, then $\alpha \cup \beta$ does too.

$\Rightarrow$ <u>gives $\cup$ on relative chains</u>.

To prove commutativity, we'll make use of the following lemma:

<u>Lemma</u>: let $T: X\times Y \longrightarrow Y \times X$ be the factor reversing map $T(x,y) = (y,x)$.

& for chain complexes $C_\bullet, D_\bullet$, let
$$\tau: C_\bullet \otimes D_\bullet \longrightarrow D_\bullet \otimes C_\bullet$$
$$(c,d) \longmapsto (-1)^{\deg(c)\deg(d)} d \otimes c$$
factor-reversing chain map.

Then the following diagram is homotopy commutative.

$$\begin{tikzcd}
C_\bullet(X\times Y) \arrow[r, "\theta_{(X,Y)}"] \arrow[d, "T_\#"] & C_\bullet(X)\otimes C_\bullet(Y) \arrow[d, "\tau"] \\
C_\bullet(Y\times X) \arrow[r, "\theta_{(Y,X)}"] & C_\bullet(Y)\otimes C_\bullet(X)
\end{tikzcd}$$

i.e., $\exists\ H: C_\bullet(X\times Y)\to (C_\bullet(Y)\otimes C_\bullet(X))_{\bullet+1}$ w/ $\partial H + H\partial = \tau\circ\theta - \theta\circ T_\#$.

Pf: Consider $\tau\circ\theta\circ T_\#$ and $\theta : C_\bullet(X\times Y)\to C_\bullet(X)\otimes C_\bullet(Y)$. These are both natural maps, chain maps, do the same thing in degree 0 $\Longrightarrow$ (Thm last time, using acyclic models) $\tau\circ\theta\circ T_\#$ and θ are chain homotopic via some chain homotopy G.

$$\Rightarrow \tau\circ\theta\circ\overbrace{T_\#\circ T_\#}^{\text{id b/c } T\circ T=\text{id}}\ \underset{G\circ T_\# =: H \text{ (chain homotopy)}}{\simeq}\ \theta\circ T_\#. \qquad \boxtimes.$$

Now, this implies:

$$C_\bullet(X)\xrightarrow{\Delta_\#} C_\bullet(X\times X)\xrightarrow{\theta} C_\bullet(X)\otimes C_\bullet(X)\xrightarrow[\text{factor reverse}]{\tau} C_\bullet(X)\otimes C_\bullet(Y)$$

is chain homotopic to

$$\begin{tikzcd}
C_\bullet(X) \arrow[r, "\Delta_\#"'] \arrow[rr, bend right, "(T\circ\Delta)_\# \overset{!}{=} \Delta_\#"'] & C_\bullet(X\times X) \arrow[r, "T_\#"'] & C_\bullet(X\times X) \arrow[r, "\theta"] & C_\bullet(X)\otimes C_\bullet(X)
\end{tikzcd}$$

, i.e., to

← why? $X\xrightarrow{\Delta} X\times X\xrightarrow{T} X\times X$

$x\longmapsto (x,x)\longmapsto (x,x)$.

$$C_\bullet(X)\xrightarrow{\theta\circ\Delta_\#} C_\bullet(X)\otimes C_\bullet(X)$$ (the usual coproduct)

Dualizing, we see that for $\alpha,\beta\in C^\bullet(X)$ (R-coeffs.), $a\in C_\bullet(X)$.

$$\alpha\smile\beta\,(a) := \overbrace{\alpha\otimes\beta}^{\times}\,(\theta(\underbrace{\Delta_\# a}_{\wedge\ C_\bullet(X)\otimes C_\bullet(X)}))\ .\ \boxed{\simeq}\ \alpha\otimes\beta\,(\tau\circ\theta\circ\Delta_\#(a))$$

implicitly applying α to first factor, β to second, & multiplying result.

chain homotopic via dualizing above

$$(-1)^{\deg(\beta)\deg(\alpha)}\beta\otimes\alpha\,(\theta\circ\Delta_\#(a)) = (-1)^{\deg(\alpha)\deg(\beta)}\beta\smile\alpha\ (a). \qquad \boxtimes.$$

We could verify (2)+(3) similarly, but in order to verify on the chain level, & for computational purposes, we'll now introduce a particular concrete model of $\smile$ on chain level.

Explicit formula for Θ: Alexander-Whitney map (last week, we hinted at a similar combinatorial angle of $EZ: C_\bullet(X)\otimes C_\bullet(Y) \to C_\bullet(X\times Y)$).

Let $\Delta^n = [e_0, \dots, e_n]$ be the standard simplex. For any $0 \le p \le n$, $0 \le q \le n$,

Define the front p-face of Δ^n to be $[e_0, \dots, e_p]$ inside $[e_0, \dots, e_n]$, or via

$$\begin{aligned} f_p : \Delta^p &\hookrightarrow \Delta^n \\ e_0 &\longmapsto e_0 \\ &\vdots \\ e_p &\longmapsto e_p . \end{aligned}$$

Define the back q-face of Δ^n to be $\overset{\Delta^q}{\overset{\shortparallel}{[e_{n-q}, \dots, e_n]}} \hookrightarrow \Delta^n$.

$$\begin{aligned} g_q : \Delta^q &\hookrightarrow \Delta^n \\ e_0 &\longmapsto e_{n-q} \\ &\vdots \\ e_q &\longmapsto e_n \end{aligned}$$

e.g., $\Delta^2 :=$

Using this, let's define an explicit version of Θ,

$$\Theta_{AW} : C_n(X\times Y) \longrightarrow \big(C_\bullet(X)\otimes C_\bullet(Y)\big)_n = \bigoplus_{i=0}^{n} C_i(X) \otimes C_{n-i}(Y)$$

"Alexander-Whitney"

by:

$$\left\{ \underset{\underset{(\pi_X\sigma,\, \pi_Y\sigma)}{\shortparallel}}{\sigma} = \Delta^n \to X\times Y \right\} \longmapsto \sum_{i=0}^{n} \underbrace{(\pi_X \sigma \circ f_i)}_{\underset{\underset{C_i(X)}{\cap}}{\pi_X\sigma|_{[e_0,\dots,e_i]}}} \otimes \underbrace{(\pi_Y \sigma \circ g_{n-i})}_{\underset{\underset{C_{n-i}(Y)}{\cap}}{\pi_Y\sigma|_{[e_i,\dots,e_n]}}}$$

(q: check signs of above)

Lemma: Θ_{AW} is natural in X, Y, is a chain map, and coincides w/ the usual definition of Θ in degree 0 ($(x,y) \longmapsto x\otimes y$).

(therefore, by acyclic models, $\theta_{AW} \simeq$ any other θ).

Pf idea: Naturality is straightforward, & explicitly need to compute

$$\partial \theta_{AW}(\sigma) \overset{?}{=} \theta_{AW}(\partial \sigma). \qquad \text{(exercise)}.$$

Using this, define $\theta_{AW} \circ \Delta_{\#}$ (chain model of homological coproduct):

$$\{\tau : \Delta^n \to X\} \xmapsto{\Delta_{\#}} \{\Delta_{\#}\tau = (\tau, \tau) : \Delta^n \to X \times X\} \xmapsto{\theta_{AW}} \sum_{i=0}^{n} \tau|_{[e_0, \ldots, e_i]} \otimes \tau|_{[e_i, \ldots, e_n]}.$$

and $\alpha \cup \beta$ can be given the model:

$$\underset{\deg p}{\alpha} \cup \underset{\deg q}{\beta} (\underset{\deg r = p+q}{\tau}) = \alpha \otimes \beta (\theta_{AW} \circ \Delta_{\#}(\tau))$$

$$= \alpha \otimes \beta \Big(\sum_{i=0}^{r} \tau|_{[e_0, \ldots, e_i]} \otimes \tau|_{[e_i, \ldots, e_n]} \Big)$$

α (i-simplex) $= 0$ unless $i = \deg(\alpha)$.

$$:= (-1)^{pq} \alpha(\tau|_{[e_0, \ldots, e_p]}) \cdot \beta(\tau|_{[e_p, \ldots, e_{n=p+q}]}).$$

(b/c $f \otimes g (a \otimes b) := (-1)^{\deg(g)\deg(a)} f(a)\, g(b)$).

2/3/2021

Last time: defined cup product via

$$\begin{tikzcd} H^{\bullet}(X) \otimes H^{\bullet}(X) \arrow[r, "\times"] \arrow[rr, bend right, "\cup"'] & H^{\bullet}(X \times X) \arrow[r, "\Delta^*"] & H^{\bullet}(X) \end{tikzcd}$$

$[\text{mult} \circ \theta^*]$, $\big(C^{\bullet}(X) \otimes C^{\bullet}(X) \xrightarrow{\text{mult}, \otimes} \text{Hom}(C_{\bullet}(X) \otimes C_{\bullet}(X), R) \xrightarrow{\theta^*} C^{\bullet}(X \times X)\big)$

- Using an explicit chain model for θ, called θ_{AW}, we gave an explicit co-chain model for $\cup$:

$$\underset{\deg p}{\alpha} \cup \underset{\deg q}{\beta} \, (\underset{\deg r = p+q}{\tau}) := (-1)^{pq} \alpha\big(\overset{\tau \text{ restr. to front } p\text{-face}}{\tau|_{[e_0,\dots,e_p]}}\big) \cdot \beta\big(\overset{\tau \text{ restr. to back } q\text{-face.}}{\tau|_{[e_p,\dots,e_{n=p+q}]}}\big)$$

(b/c $f \otimes g\,(a \otimes b) := (-1)^{\deg(g)\deg(a)} f(a)\, g(b)$).

It turns out this co-chain model for cup product is associative on the chain level & unital, ($\alpha \cup 1 = 1 \cup \alpha = \alpha$) on chain level (by direct computation), verifying associativity + unitality properties of $[\cup]$ (for any chain level model) on cohomology.

The cup product on relative co-chains:

X space, $A, B \underset{\text{open}}{\subset} X$.

We know $C^n(X,A) = \mathrm{Ann}(C_n(A)) \subset \mathrm{Hom}(C_n(X), R)$ (w/ R-coeffs.)

If $\phi \in C^p(X,A)$, $\psi \in C^q(X,B)$, where $p+q=n$,

$$\phi \cup \psi(\underset{C_n}{\sigma}) = \pm\, \phi(\sigma|_{[e_0,\dots,e_p]}) \cdot \psi(\sigma|_{[e_p,\dots,e_{n=p+q}]})$$

is zero if $\mathrm{im}(\sigma) \subset A$ or if $\mathrm{im}(\sigma) \subset B$ entirely, so $\phi \cup \psi \in \mathrm{Ann}(C_n(A) + C_n(B))$,

(b/c then its front p-face is too) (b/c then its back face is too).

we'll use the shorthand $\mathrm{Ann}(C_n(\text{"}A+B\text{"})) \overset{\text{def}}{=} \mathrm{Ann}(C_n(A) + C_n(B))$.

$$\begin{tikzcd} \mathrm{Ann}(C_n(\text{"}A+B\text{"})) \arrow[d, phantom, "\parallel"] & \\ C^n(X, \text{"}A+B\text{"}) \arrow[r, phantom, "\overset{\text{not nec.}}{\neq}"] & C^n(X, A \cup B) \arrow[l, bend left, "\exists \text{ natural map (incl.)}"] \end{tikzcd}$$

annihilates simplices in A or in B. annihilates simplices in $A \cup B$

We'll also abbreviate $C_n(A+B) := C_n(A) + C_n(B)$ (sum in $C_n(X)$). We have a natural inclusion.

$i : C_n(A+B) \hookrightarrow C_n(A \cup B)$, which we note (by prev. semester) induces an iso. on homology $H_n(C_\bullet(A+B)) \xrightarrow{\cong} H_n(C_\bullet(A \cup B))$.

($Y = A \cup B$ w/ cover $\{A, B\}$)

(more generally, barycentric subdiv. $\Rightarrow$ for any Y w/ a cover $\mathcal{U} = \{U_i\}$,

$$\underset{\text{chains supported in some } U_i}{C_\bullet^{\mathcal{U}}}(Y) \xrightarrow[\text{on } H.]{\cong} C_\bullet(Y)).$$

$\Rightarrow$ induces an iso. on cohomology $H^n(A \cup B) \xrightarrow{\cong} H^n(\text{"}A+B\text{"})$.

By comparing LES of pair $(X, A\cup B)$ w/ pair $(X, ``A+B")$ in cohomology, (exercise) we can deduce that the canonical map $C^\bullet(X, A\cup B) \xrightarrow{k} C^\bullet(X, ``A+B")$ induces a cohomology iso,

$$[k]: H^n(X, A\cup B) \xrightarrow{\cong} H^n(X, ``\underset{\Downarrow}{A}+B")\,.$$

$$(-) \;\; 1\cdot \dashrightarrow [\phi\cup\psi]$$

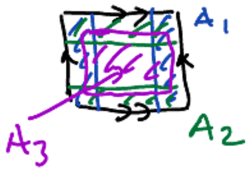
Cor: Get a cup product map $[\cup]: (= [k]^{-1}\circ(\cup)) \; : H^p(X,A) \otimes H^q(X,B) \longrightarrow H^{n=p+q}(X, A\cup B)$.

Exercise on HW (using this): show that if X is covered by m acyclic open sets then all m-fold cup products of pos. degree classes are zero.

e.g.,

$\Rightarrow$ any α, β pos. degree coh.

$\Rightarrow \alpha\cup\beta = 0$.



$\Rightarrow$ In T^2 given $\alpha_1, \alpha_2, \alpha_3$ pos. degree, $\alpha_1 \cup \alpha_2 \cup \alpha_3 = 0$ in cohomology.

conversely, we'll compute $H^\bullet(T^2)$ has a non-trivial cup product of degree 1-classes

$\Rightarrow T^2 \neq A\cup B$, A, B contractible.

Compatibility with cross product

X, Y spaces, R coefficient ring (implicit), have

- $\times : H^\bullet(X) \otimes H^\bullet(Y) \longrightarrow H^\bullet(X\times Y)$
- $\Delta_X : X \to X\times X$, $\Delta_Y : Y \to Y\times Y$, $\Delta_{X\times Y} : X\times Y \longrightarrow X\times Y\times X\times Y$.

Obs: $\Delta_{X\times Y} = T\circ(\Delta_X, \Delta_Y)$ where $T: X\times X\times Y\times Y \longrightarrow X\times Y\times X\times Y$ swaps 2nd + 3rd factors.

Lemma: $(\overset{H^\bullet(X)}{\alpha_1} \times \overset{H^\bullet(Y)}{\beta_1}) \cup (\alpha_2 \times \beta_2) = (-1)^{\deg(\alpha_2)\deg(\beta_1)} (\alpha_1\cup\alpha_2) \times (\beta_1 \cup \beta_2)$

Pf: $LHS = \Delta^*_{X\times Y}\big((\alpha_1\times\beta_1)\times(\alpha_2\times\beta_2)\big)$

$\overset{\text{obs}}{=} (\Delta_X, \Delta_Y)^* T^* (\alpha_1\times\beta_1 \times \alpha_2\times\beta_2$

erase parentheses b/c $\times$ is assoc. (on cohomology) (on chain level, associates may cost chain homotopy - unless using θ_{AW}).

$$= (-1)^{\deg(\alpha_2)\deg(\beta_1)} (\Delta_X, \Delta_Y)^* (\alpha_1 \times \alpha_2 \times \beta_1 \times \beta_2)$$

$= RHS.$ $\square$

<u>Cor</u>: The Künneth isomorphism (which holds when one of X, Y is of 'finite type'):

$$H^\bullet(X) \otimes H^\bullet(Y) \xrightarrow{\times} H^\bullet(X \times Y)$$

is a <u>ring iso</u>.

(LHS has a ring str. by $(x \otimes y) \cdot (x' \otimes y') := (-1)^{\deg(y)\deg(x')} (x \smile x') \otimes (y \smile y')$.)

$\pi_X : X \times Y \to X$
$\pi_Y : X \times Y \to Y.$

<u>Cor</u>: ($\times$ in terms of $\smile$). For any $\alpha \in H^\bullet(X)$, $\beta \in H^\bullet(Y)$

$$\boxed{\alpha \times \beta} \underset{\substack{\text{Lemma} \\ \text{(\& unitality)}}}{=} (\alpha \times 1_Y) \smile (1_X \times \beta) \underset{\substack{\text{last} \\ \text{time}}}{=} \boxed{(\pi_X^* \alpha) \smile (\pi_Y^* \beta)}$$

<u>Rmk</u>: RHS is how Hatcher defines $\times$, at least initially.

—

<u>Example</u>: (1) compute $H^\bullet(S^2 \times S^4)$.

We have $H^\bullet(S^{2k}) = \begin{cases} \mathbb{Z} & \deg 2k \\ \mathbb{Z} & \deg 0 \\ 0 & \text{else} \end{cases}$ (by UCT).

Let's denote the degree $2k$ generator by α_{2k}. Note $\alpha_{2k} \smile \alpha_{2k} = 0$ b/c $H^{4k}(S^{2k}) = 0$.

So, as a ring, $H^\bullet(S^{2k}) \cong \mathbb{Z}[\alpha_{2k}]/\alpha_{2k}^2$, degree$(\alpha_{2k}) = 2k$.
$|\alpha_{2k}|$

mean $\bigoplus_i H^i(S^{2k})$

Therefore by Künneth ring isomorphism (works over $\mathbb{Z}$ b/c $H^\bullet(S^{2k})$ free finite type),

$$H^\bullet(S^2 \times S^4) \cong \mathbb{Z}[\alpha_2]/\alpha_2^2 \otimes \mathbb{Z}[\alpha_4]/\alpha_4^2 \cong \mathbb{Z}[\alpha, \beta]/\alpha^2, \beta^2,$$

$|\alpha| = 2$, $|\beta| = 4$.

Note $\alpha \cdot \beta$ generates in degree 6.

$(\alpha = pr_{S^2}^* \alpha_2,\ \beta = pr_{S^4}^* \alpha_4)$.

(2) $T^n = (S^1)^n$. By same reasoning as above $H^*(S^1;\mathbb{Z}) \cong \mathbb{Z}[\theta]/\theta^2$ $\quad |\theta| = 1$.
$\cong \mathbb{Z}[\theta]$

(modding by θ^2 is redundant if we're imposing graded commutativity over $\mathbb{Z}$:
$$\theta \cup \theta = (-1)^{\deg(\theta)\deg(\theta)}\theta\cup\theta = -\theta\cup\theta \underset{\text{over } \mathbb{Z}}{\Rightarrow} \theta\cup\theta = 0)$$

Therefore $H^*(T^n) \cong \mathbb{Z}[\theta_1,\dots,\theta_n]$ "exterior algebra in n-variables"
(w/ $\theta_i^2=0$ implicit). (sometimes $\mathbb{Z}|\theta_1,\dots,\theta_n|$ or ___)

each $|\theta_i| = 1$ $\quad \theta_i = (\pi_i^*)(\theta)$.

$j \neq i$: $\theta_i\cdot\theta_j = -\theta_j\cdot\theta_i$. (by graded commutativity).

Since $\theta_1\cup\cdots\cup\theta_n \neq 0 \overset{\text{(exercise above)}}{\Rightarrow}$ any "acyclic cover" of T^n has cardinality $> n$.

Ex: $S^2\vee S^4$. know (by UCT): $H^k(S^2\vee S^4) = \begin{cases} \mathbb{Z} & \deg 4, \text{ gen. by } j_{S^4}^*\alpha_4 = x_2 \\ \mathbb{Z} & \deg 2, \text{ gen. by } j_{S^2}^*\alpha_2 = x_1 \\ \mathbb{Z} & \deg 0 \end{cases}$

$j_{S^2}: S^2\vee S^4 \xrightarrow[\text{projection (collapse } S^4 \text{ to point)}]{} S^2 \qquad j_{S^4}: S^2\vee S^4 \to S^4$

check: $j_{S^2}^*: H^2(S^2) \to H^2(S^2\vee S^4)$ is an isomorphism (exercise)

Q: is there a relation in H^4 between x_1^2 and x_2?

By naturality, $x_1\cup x_1 = j_{S^2}^*\alpha_2 \cup j_{S^2}^*\alpha_2 \underset{\text{naturality}}{=} j_{S^2}^*(\underbrace{\alpha_2\cup\alpha_2}_{H^4(S^2)=0.}) = j_{S^2}^*(0) = 0$.

So, no.

Hence $H^*(S^2\vee S^4) \cong \mathbb{Z}[\underset{\deg 2}{x_1}, \underset{\deg 4}{x_2}]/\overset{\text{by above}}{x_1^2}, \overset{\text{b/c } H^8=0}{x_2^2}, \overset{\text{b/c } H^6=0}{x_1x_2}$..

Important examples:

$\underset{\text{over } R=\mathbb{Z}/2}{\mathbb{RP}^n}$, $\underbrace{\mathbb{CP}^n, \mathbb{HP}^n}_{\text{over } R=\mathbb{Z}}$, $\quad n\in\mathbb{N}\cup\{\infty\}$. (via e.g., $\mathbb{CP}^\infty = \bigcup_{n}^{n=\infty}\mathbb{CP}^n$, where $\mathbb{CP}^1\hookrightarrow\mathbb{CP}^2\hookrightarrow\cdots$)

(+UCT)

We know from studying cellular chain complexes that

$H^k(\mathbb{RP}^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & k=0,\dots,n \\ 0 & \text{else.} \end{cases}$, similarly $H^k(\mathbb{CP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0,2,\dots,2n \\ 0 & \text{else.} \end{cases}$

$H^k(\mathbb{HP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0,4,\dots,4n \\ 0 & \text{else.} \end{cases}$

Write. $h \in H^1(\mathbb{RP}^n;\mathbb{Z}/2)$ generator of H^1.

(since over $\mathbb{Z}/2$, $h\cup h$ need not be zero).

Thm: $h^k := \underbrace{h\cup\dots\cup h}_{k \text{ times}}$ is a generator for $H^k(\mathbb{RP}^n;\mathbb{Z}/2)$, for any $k\le n$,

i.e., $\boxed{H^\bullet(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2\,[h]/h^{n+1}.}$ (if $n<\infty$) truncated polynomial alg.

$|h|=1.$

Thm: If $h\in H^2(\mathbb{CP}^n;\mathbb{Z})$ is a generator for H^2, then h^k generates $H^{2k}(\mathbb{CP}^n;\mathbb{Z})$ for all $k=1,\dots,n$.

$\Rightarrow H^\bullet(\mathbb{CP}^n) \cong \mathbb{Z}[h]/h^{n+1}.$ $\quad |h|=2.$

similarly for $\mathbb{HP}^n$, $|h|=4$.

We'll prove these theorems later, as a consequence of other results (e.g., Poincaré duality)

But we can already explore some consequences:

- Observe that $H^i(\mathbb{CP}^3;\mathbb{Z}) \cong H^i(S^2\times S^4;\mathbb{Z})$ in every degree (similarly on H_i).

  ($\mathbb{Z}$ in degree $0,2,4,6$, 0 otherwise)

  However, the ring structures on $H^\bullet$ are different:

$H^\bullet$	$\mathbb{Z}[h]/h^4$	vs.	$\mathbb{Z}[\alpha,\beta]/\alpha^2,\beta^2$	$\lvert\alpha\rvert=2, \lvert\beta\rvert=4.$
	$\lvert h\rvert=2$			
	$\mathbb{CP}^3$		$S^2\times S^4$	

are not isomorphic rings. so $\mathbb{CP}^3 \not\simeq S^2\times S^4$ · up to homotopy equivalence

(e.g., any ~~iso~~ homotopy equivalence would send h to $\pm\alpha$, but $h^2\neq 0$, and $\alpha^2=0$ $\Rightarrow\!\Leftarrow$).

- Look at $\mathbb{CP}^2$ vs. $S^2 \vee S^4$.

Note: $\mathbb{CP}^2 = \underbrace{e^0 \cup e^2}_{\mathbb{CP}^1 = S^2} \cup e^4$ $\qquad$ $S^2 \vee S^4 = \underbrace{e^0 \cup e^2}_{S^2} \cup e^4$

If $\mathbb{CP}^2 \underset{h.e.}{\not\simeq} S^2 \vee S^4$, we conclude the attaching maps $f_{\mathbb{CP}^2} : \partial e^4 = S^3 \longrightarrow \overbrace{S^2}^{\text{3-skeleton}}$ and

$$f_{S^2 \vee S^4} : \partial e^4 = S^3 \xrightarrow[\substack{\text{const.} \\ \text{at } e^0}]{} S^2$$

cannot be homotopic.

Let's check H^* rings: $H^*(\mathbb{CP}^2) \cong \mathbb{Z}[h]/h^3 \quad |h| = 2.$

$$H^*(S^2 \vee S^4) \not\cong \mathbb{Z}[\alpha, \beta]/\alpha^2, \beta^2, \alpha\beta \quad |\alpha| = 2, |\beta| = 4$$

So The attaching map: $S^3 \to S^2$ ('Hopf' map) used to construct $\mathbb{CP}^2$ ~~is~~ represents a ~~[illegible]~~ non-trivial homotopy class. $(\Rightarrow \pi_3(S^2) \neq 0)$.

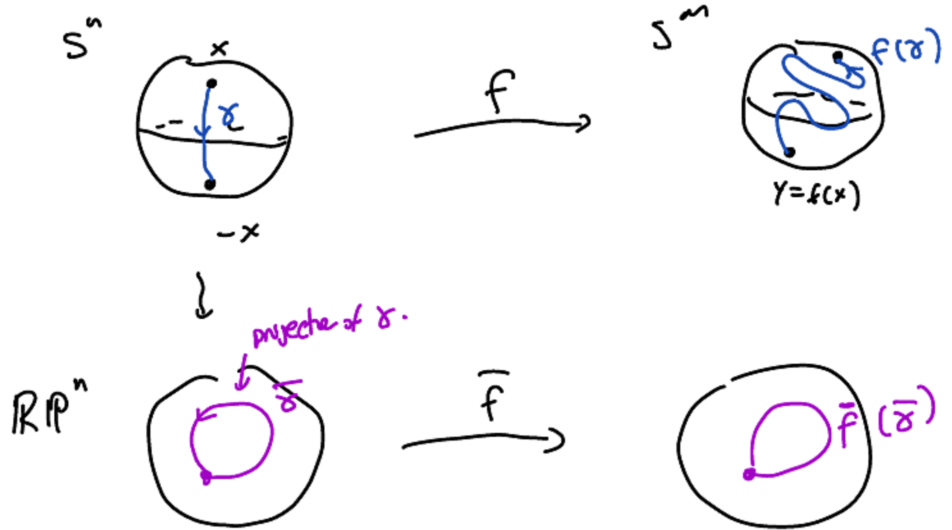
$$\pi_k(X) = [S^k, X]_{\text{based}}$$

(of $H^*(\mathbb{RP}^n)$)

Application: Say we have an odd map $f : \underset{\text{unit spheres}}{S^n \to S^m}$, i.e., $f(-x) = \overset{\text{antipodal.}}{-f(x)}$.

$\Rightarrow$ get a map $\bar{f} : \mathbb{RP}^n \longrightarrow \mathbb{RP}^m$ $\qquad$ $(\mathbb{RP}^k = S^k/\{\pm 1\})$.

$\{\pm x\} \longmapsto \{\pm f(x)\}$.



$\bar{f}(\bar{\gamma})$ is a generator of $H_1(\mathbb{RP}^m; \mathbb{Z}/2), \cong \mathbb{Z}/2$

(b/c it lifts under $S^m \xrightarrow{2:1} \mathbb{RP}^m$ to a path between lifts of x, hence is a non-trivial element of $\pi_1(\mathbb{RP}^m)$, (f 'odd' if $m=1$))

generator of $H_1(\mathbb{RP}^n; \mathbb{Z}/2)$

$\Rightarrow \bar{f}_* : H_1(\mathbb{RP}^n; \mathbb{Z}/2) \xrightarrow{\cong} H_1(\mathbb{RP}^m; \mathbb{Z}/2)$

$\Rightarrow \bar{f}^* : H^1(\mathbb{RP}^m; \mathbb{Z}/2) \xrightarrow{\cong} H^1(\mathbb{RP}^n; \mathbb{Z}/2)$

UCT

$$\overset{\rotatebox{90}{$\in$}}{h_{\mathbb{RP}^m}} \text{ generator} \longmapsto h_{\mathbb{RP}^n} \text{ generator.}$$

By naturality of cup product

$$\bar{f}^*(h_{\mathbb{RP}^m}^k) = \bar{f}^*(h_{\mathbb{RP}^m})^k = h_{\mathbb{RP}^n}^k$$

If we take $k = m+1$

$$\Rightarrow \underset{\underset{0.}{\|}}{\bar{f}^*(h_{\mathbb{RP}^m}^{m+1})} = \underset{\underset{0 \text{ if } m<n.}{\nparallel}}{h_{\mathbb{RP}^n}^{m+1}}$$

<u>Cor:</u> $m \geq n$.

<u>Cor:</u> $\nexists$ an odd map $S^n \to S^m$ $m < n$.

<u>Cor:</u> (<u>Borsuk-Ulam theorem</u>): Given any continuous $g: S^n \to \mathbb{R}^n$, $\exists\, x \in S^n$ with $g(x) = g(-x)$.

<u>Pf:</u> If not, define $f: S^n \to S^{n-1}$ by

$$\frac{g(x) - g(-x)}{\|g(x) - g(-x)\|}, \text{ \& note } -f(x) = f(-x) ⨳ \quad \boxtimes.$$

Last remark about cup product:

The explicit co-chain formula we wrote down for $\cup$ (using Θ_{AW}) has another advantage: it's simplicial, i.e., defined on the level of simplicial co-chains as well.

In particular, the map (for a simplicial or Δ-complex X)

$$C^\bullet_{\text{Simp}}(X) \leftarrow C^\bullet_{\text{Sing}}(X) \quad \Big(\text{dual to } C_\bullet^{\text{Simp}}(X) \longrightarrow C_\bullet^{\text{Sing}}(X)$$

simplex in $X \longmapsto$ same simplex, thought of as a singular simplex)

intertwines $\cup$ products (defined using "front + back face").

~~W~~ Can use this to make explicit computations ~~of~~ of $\cup$ in ~~sing~~ simplicial co-chains.

(exercise).

Next week: cap product ($H^\bullet$ acts on $H_\bullet$), Poincaré duality.