The cap product

We've introduced $H^\bullet(X)$ & showed it has structure of (graded) comm. ring, via cup product.

It turns out that $H_\bullet(X)$ has the structure of a (graded) module over $H^\bullet(X)$, via an operation called the cap product.

Fundamentally, on the chain level, the cap product is induced by the same information as the cup product, namely the homological coproduct

$$\Delta : C_\bullet(X) \longrightarrow C_\bullet(X) \otimes C_\bullet(X)$$

defined by $\Delta := \Theta \circ \Delta_\#$

the map $C_\bullet(X) \longrightarrow C_\bullet(X\times X)$ induced by $\Delta : X \to X\times X$

the inverse to EZ: $C_\bullet(X\times X) \xrightarrow{\simeq} C_\bullet(X)\otimes C_\bullet(X)$.

Def'n: Given $\alpha \in C^p(X)$, $\beta \in C_q(X)$,

define $\alpha \frown \beta := id\otimes\alpha(\underbrace{\Delta\beta})$

lives in $(C_\bullet(X)\otimes C_\bullet(X))_q := \bigoplus_{i+j=q} C_i(X)\otimes C_j(X)$

$\Delta\beta := \sum_{i+j=q} (\Delta\beta)_{i,j}$

recall $\alpha : C_p(X) \to R$,

extend to $\alpha : C_\bullet(X) \to R$ by saying $\alpha(C_i(X)) = 0$ for $i \neq p$.

i.e., $\alpha\frown\beta = id\otimes\alpha(\Delta\beta_{q-p,p}) \in C_{q-p}(X)$,

b/c $id\otimes\alpha$ is only non-zero on this piece.

If we use the Alexander-Whitney model of homological coproduct

$$\Delta_{AW} := \Theta_{AW}\circ\Delta_\# : \underset{\text{degree } q}{\sigma} \longmapsto \sum_{i+j=q} \sigma|_{[e_0,\dots,e_i]} \otimes \sigma|_{[e_i,\dots,e_{i+j=q}]},$$

front i-face

back j-face

we see that up to chain homotopy the cap product can be given the following chain-level model; linearly extend the following formula over all chains

∃ a sign to apply α to this term

$$\alpha^p \frown \sigma_q := (id\otimes\alpha^p)\left(\sigma|_{[e_0,\dots,e_{q-p}]} \otimes \sigma|_{[e_{q-p},\dots,e_{q-p+p=q}]}\right)$$

front q-p face

back p face.

$C^p(X;R)$

$C_q(X;R)$ generator $\sigma : \Delta^q \to X$.

$$= (-1)^{p(q-p)}\, \sigma|_{[e_0,\dots,e_{q-p}]} \cdot \underbrace{\alpha(\sigma|_{[e_{q-p},\dots,e_p]})}_{R.}$$

Prop: For any Θ (not just Θ_{AW}), the cap product satisfies the property of being a chain map

$$\underset{\substack{\uparrow \\ \text{co-chains on} \\ X \text{ w/ degrees}}}{C^{-\bullet}(X)} \otimes C_\bullet(X) \xrightarrow{\frown} C_\bullet(X) \quad \text{(of degree } 0\text{)}.$$

co-chains on X w/ degrees negated is a chain complex, in sense that δ decreases (-degree) by 1.

$$\underbrace{C^p(X)}_{} \otimes C_q(X) \xrightarrow{\frown} C_{q-p}(X).$$

↑ think of this as a degree $-p$ element of $C^{-\bullet}(X)$, then degrees are additive under cup product.

Namely, $\boxed{\partial(\alpha \frown \sigma) = \delta\alpha \frown \sigma + (-1)^p \alpha \frown \partial\sigma}$ (where $\alpha \in C^p(X)$)..

Hence, there is an induced map $H^\bullet(X) \otimes H_\bullet(X) \xrightarrow{\frown} H_\bullet(X)$, which negating degrees of cohomology groups, is a graded map.

Prop: $\frown$ on homological level is independent of choice of Θ (not hard to see), and satisfies:

(1) $\underset{\underset{H^0(X)}{\cap}}{1} \frown \gamma = \gamma$ for all $\gamma \in H_p(X)$

(2) If $\varepsilon: X \to pt$ induces $\varepsilon_*: H_0(X) \xrightarrow{\cong} H_0(pt) = R$ ('augmentation') (note $1 = \varepsilon^*(1)$).,
and $[\alpha] \in H^p(X)$, $[\tau] \in H_p(X)$, then $\underset{\underset{R}{\cap}}{\varepsilon_*}(\underbrace{[\alpha] \frown [\tau]}_{\underset{H_0(X)}{\cap}}) = \underset{\underset{\in C^p(X;R) = \mathrm{Hom}_R(C_p(X), R).}{\uparrow}}{\alpha}(\tau)$

(3) $(\alpha \smile \beta) \frown \gamma = \alpha \frown (\beta \frown \gamma)$

(note an action of R on M is a module action $\Rightarrow (r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$).

(4) If $\lambda: X \to Y$ is a map of top. spaces, $\alpha \in H^\bullet(Y)$, $\beta \in H_\bullet(X)$, then:

$$\underbrace{\alpha \frown \lambda_*(\beta)}_{\underset{H_\bullet(Y)}{\cap}} = \lambda_*(\underbrace{\lambda^*\alpha}_{\underset{H^\bullet(X)}{\cap}} \frown \beta).$$

Interpretation: Since $\lambda^*: H^\bullet(Y) \to H^\bullet(X)$ is a ring map, $H_\bullet(X)$ (a module over $H^\bullet(X)$) becomes via λ^* a module over $H^\bullet(Y)$ via the action "applying λ^* then $\frown$".

(4) is then stating that $H_\bullet(X) \xrightarrow{\lambda_*} H_\bullet(Y)$ is a map of $H^\bullet(Y)$-modules (w.r.t. this module action of $H^\bullet(Y)$ on $H_\bullet(X)$ induced by λ^*).

Proofs of prop: straightforward: (4) is an immediate consequence of naturality (& holds on chain level for any Θ) and (1)-(3) can be

checked on chain level for the Alexander-Whitney model of $\cap$, hence hold homologically for any model.
(e.g., (3) follows from "Δ is co-associative") (exercise).

$C^*(X)$

Rmks: If $A \subset X$, and c is a chain in A, then $\alpha \cap c$ is a chain in A too, for any $\overset{\in}{\alpha}$,
so get $\cap : C^p(X) \otimes C_n(X,A) \longrightarrow C_{n-p}(X,A)$, inducing a homology level cap product.

$$\underset{\substack{\cup \\ Ann(C_\bullet(A)) \\ \shortparallel}}{C^\bullet(X)} \otimes C_\bullet(X) \longrightarrow C_\bullet(X)$$

$Ann(C_\bullet(A)) \otimes C_\bullet(X) \to C_\bullet(X)$
sends
$Ann(C_\bullet(A)) \otimes C_\bullet(A) \to 0$, hence get a map
$Ann(C_\bullet(A)) \otimes \frac{C_\bullet(X)}{C_\bullet(A)} \to C_\bullet(X)$.

Exercises:

- Also get

$$C^p(X,A) \otimes C_n(X,A) \longrightarrow C_{n-p}(X)$$

idea: elements here are ~~ae~~ zero on $C_\bullet(A) \subset C_\bullet(X)$, so unaffected by adding chains on A.

- More generally, if $A, B \underset{\text{open}}{\subset} X$, $\left(C_\bullet(\text{"}A+B\text{"}) \simeq C_\bullet(A \cup B) \right)$,
means $C_\bullet(A)+C_\bullet(B)$ (in $\underline{C_\bullet(X)}$).

get $C^p(X,A) \otimes C_n(X, \text{"}A+B\text{"}) \xrightarrow{\cap} C_{n-p}(X,B)$, inducing
meaning $\dfrac{C_n(X)}{C_n(A)+C_n(B)}$

$$H^p(X,A) \otimes H_n(X, A \cup B) \to H_{n-p}(X,B),$$

etc. .

Orientations:

On a finite-dim'l vector space V^n ($n := \dim(V)$), an orientation is a choice of ordered basis up to equivalence, where bases

$(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ are equivalent if the map taking one to another has positive determinant.

To generalize to the case $n=0$ in a uniform way, we might equivalently say an orientation is a choice of non-zero element of $\Lambda^{\dim(V)} V$ (given $\Lambda^0 \{0\} \cong \mathbb{R}$ canonically), up to the equivalence relation of positive scaling.

$\Rightarrow$ two possible choices. (even in dim 0, where the two choices are $\{+,-\}$).

Denote by $o(V)$ (or $or(V)$) the set of orientations on V (two elements, but not canonically identified w/ +/- unless dimension is 0. On the other hand the operation called "orientation reversal" induces a free $\mathbb{Z}/2$ action on $o(V)$ — so $o(V)$ is a $\mathbb{Z}/2$ - torsor).

If M^n smooth manifold: (say $M \hookrightarrow \mathbb{R}^N$)

Then at any p, have a tangent space T_pM (vec space of dim. n), & can pick an orientation of T_pM, $o_p \in o(T_pM)$

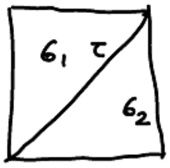
An orientation on M is really a "continuously" or smoothly" varying choice of such $\{o_p\}_{p\in M}$.
or "coherently"

Say $\{o_p\}_{p\in M}$ is 'coherent' if $\forall p \in M$, $\exists$ $U \ni p$ and a basis of vector fields $v_1, \dots, v_n$ over U with $o_q = o((v_1)_q, \dots, (v_n)_q)$ for any $q \in U$.

(in particular, given a fixed o_p, coherence condition induces a unique o_q on any such U).

How to generalize to the not nec. smooth case?

piece of M :



Idea: (old historical) Say M admits a fixed triangulation. one attempt to 'orient' M is to 'orient each top simplex'.
(meaning order the vertices for each σ): so that (coherence) for any σ_1, σ_2 sharing an edge τ, τ appears ~~sg~~ w/ opposite signs in $\partial\sigma_1$ and $\partial\sigma_2$. ($\Rightarrow$ τ cancels in $\partial(\sigma_1+\sigma_2)$).

(recall $\partial[e_0,\dots,e_n] = \sum(-1)^i [e_0,\dots,\hat{e}_i,\dots,e_n]$, signs depend on orderings)

($\Rightarrow \partial(\sum \sigma_i) = 0$ if M cpt, so expect $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$), connected).
top cycles

Problem: ad hoc, depends on triangulations.

Def: A (topological) manifold M of dimension n, denoted M^n is a space (implicitly Hausdorff, 2^{nd} countable), which is locally homeomorphic to $\mathbb{R}^n$.
(i.e., at each p $\exists$ open $U \ni p$ w/ $(U,p) \underset{\text{homeo}}{\cong} (\mathbb{R}^n, 0)$.).

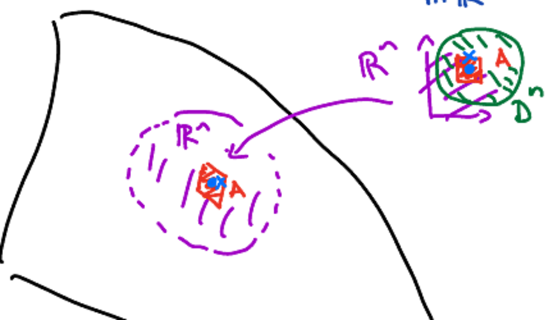
How to define a local orientation of M at x?

idea: an ordering of the vertices of this simplex should determine a 'local' orientation of M at x.

idea: such simplices live in $H_n(M, M-x)$, & an ordering determines a choice of generator.

Lem: M^n manifold, $x \in M$ any point, R any coeff grp (imagine $\mathbb{Z}$ for now). Then $H_n(M, M-\{x\}; R) \cong R$.

More generally, if $A \subset \mathbb{R}^n \xhookrightarrow{\text{open}} M$, then $H_n(M, M-A) \xrightarrow{\cong} H_n(M, M-x) \xrightarrow{\cong} R$. (for any $x \in A$)
compact convex set in $\mathbb{R}^n$



Pf: $\exists$ a closed ball $D^n \subset \mathbb{R}^n$, centered at x. containing A in its interior. Now, note there's a homotopy equivalence of pairs $D^n \xhookrightarrow{\sim} \mathbb{R}^n$ and $\partial D^n \underset{\text{incl.}}{\xhookrightarrow{\text{h.e.}}} \mathbb{R}^n - A$

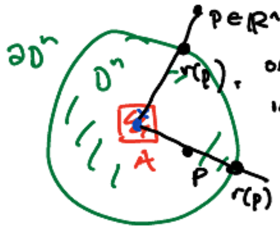
$$(D^n, \partial D^n) \underset{\text{incl.}}{\xhookrightarrow{\sim}} (\mathbb{R}^n, \mathbb{R}^n - A)$$

Q: why is this true? (exercise: need a retraction $\mathbb{R}^n - A \to \partial D^n$.

$x \in A)$

$$\begin{tikzcd} \arrow[d, "\text{(htpy equiv. by } D^n \hookrightarrow \mathbb{R}^n\text{, } S^{n-1} \hookrightarrow \mathbb{R}^n \setminus 0\text{)}"] \\ (\mathbb{R}^n, \mathbb{R}^n - x) \end{tikzcd}$$

incl. (here since other two arrows are homotopy equivalences, this one is too).



$p \in \mathbb{R}^n - A$ retracts onto the unique point $r(p)$ in ∂D^n on ray from x to p.

(check homotopy inverse to incl. : $\partial D^n \hookrightarrow \mathbb{R}^n - A$).

Hence, we have (w/ R-coeffs):

$$\begin{tikzcd} H_n(M, M-A) \arrow[rr, "\cong", "\text{incl.}"'] & & H_n(M, M-x) \\ H_n(\mathbb{R}^n, \mathbb{R}^n - A) \arrow[u, "\cong\ \text{excision}"'] \arrow[rr, "\text{incl.}\ \cong", "\text{by above}"'] & & H_n(\mathbb{R}^n, \mathbb{R}^n - x) \arrow[u, "\cong\ \text{excision (excise } M \setminus \mathbb{R}^n\text{)}"'] \\ & H_n(D^n, \partial D^n) \arrow[ul, "\cong", "\text{by above}"'] \arrow[ur, "\cong"', "\text{by above}"] \arrow[d, "\cong\ (D^n, \partial D^n)\ \text{`good pair'}"] & \\ & H_n(D^n/\partial D^n \cong S^n) \cong R. & \end{tikzcd}$$

∎.

Shorthand: $H_n(M|x; R) := H_n(M, M-x; R)$ (& leave out $R \Rightarrow$ working over $\mathbb{Z}$ unless otherwise stated).

work over $R = \mathbb{Z}$ for this def'n:

Def: A local orientation of M^n (manifold) at $x \in M$ is a choice of generator $\mu_x \in H_n(M|x) \overset{\text{non-canonically.}}{\cong} \mathbb{Z}$ (two choices of generator, since working over $\mathbb{Z}$)

An orientation on M, if it exists, is a choice of local orientations $\{\mu_x\}_{x \in M}$ which varies 'coherently' or 'continuously' in a suitable sense.

'Coherent': means that for any $x \in M$, $\exists$ a closed ball $x \in B \underset{\text{closed ball}}{\hookrightarrow} \mathbb{R}^n \hookrightarrow M$, ~~such~~ such that the induced isomorphism, for any $y \in B$,

$$\begin{tikzcd} H_n(M|B) \arrow[r, "\cong", "\text{(by lemma)}"'] \arrow[d, "\cong\ \text{incl. (by lemma)}"'] & H_n(M|x) \arrow[dl, dashed] \\ H_n(M|y) & \end{tikzcd}$$

means $H_n(M, M-B)$

sends μ_x to μ_y..

Next time: make sense of 'continuous'.

——— 2/10/2021 ———

Today: in what sense is such an orientation a "continuously varying choice"? it will be a (continuous) section of a suitable <u>bundle over M</u> (or $\mathbb{Z}/R$-modules); in this case a covering space over M.

Fix R, M^n as above.

means $H_n(M, M-x; R)$

Define
$$M_R = \coprod_{x\in M} H_n(M|x;R) = \{\alpha_x \in H_n(M|x;R),\ x\in M\};$$
in particular have $M_{\mathbb{Z}}$, $M_{\mathbb{Z}/2}$.

We can topologize M_R by, for any ball $B \subset M$ (w/ say closure $\overline{B}$ giving a closed ball in some $\mathbb{R}^n \hookrightarrow M$), and any $\alpha_B \in H_n(M|\overline{B})$, considering the sets:

$$U(\alpha_B) := \left\{ \alpha_x \in H_n(M|x;R),\ x \;\middle|\; \begin{array}{l} x \in B, \text{ and} \\ \alpha_x = \text{image of } \alpha_B \text{ under} \\ H_n(M|\overline{B}) \xrightarrow{\cong} H_n(M|x) \end{array} \right\}$$

Lem (last time). R-coeffs.

$U(\alpha_B) \subset M_R$; these give a basis for the topology we put on M_R.

There is a map $\pi : M_R \longrightarrow M$, which is continuous, and presents M_R as a (infinite-sheeted if e.g., $R=\mathbb{Z}$) covering space
$$(\alpha_x, x) \longmapsto x$$
over M (we're focusing on the cases $R = \mathbb{Z}$ or $\mathbb{Z}/2$; in general here R is discrete).

(In fact $\pi : M_R \to M$ is a <u>bundle of R-modules</u> over M: every fiber $(M_R)_x := \pi^{-1}(x) = H_n(M|x;R)$ is an R-module, and at every point x $\exists$ $U \ni x$ so that $\pi^{-1}(U) \xrightarrow{\cong} U \times \{\text{a fixed R-module, in this case } R\}$.

(in a way compatible with projections & R-module structures on each fiber)).

Recall a (implicitly continuous) section of a covering space $\widetilde{X} \xrightarrow{\pi} X$ is a (continuous) map $s : X \longrightarrow \widetilde{X}$ with $s \circ \pi = id_M$.

More generally, a section of a bundle of R-modules is defined the same way; can collect the set of sections of $Y \xrightarrow{\pi} X$

$$\Gamma(Y) := \{ s : X \to Y \mid \underline{\pi \circ s = id_X} \}.$$

obs: this is an R-<u>module too</u>:

- <u>can add</u>: $(s_1 + s_2)(x) := (x, (s_1)_x + (s_2)_x)$

in other words $s : x \longmapsto (x, s_x)$

- <u>can mult. by R</u>: $(r \cdot s)(x) := (x, r s_x)$.

<u>Re-def</u>: An orientation (or more generally an R-orientation) of M^n is

a section (implicitly continuous)

(exercise: compare Redef to original def., i.e., compare 'continuous' to 'coherently varying').

$$\begin{tikzcd} M \arrow[r, "s"] & M_{\mathbb{Z}} \\ x \arrow[r, mapsto] & \mu_x \end{tikzcd}$$

(or more generally M_R)

↗ (shorthand for (μ_x, x))

whose values μ_x at each point generate $H_n(M|x)$ (resp. $H_n(M|x; R)$).

There is a sub covering space $\widetilde{M} \subset M_{\mathbb{Z}}$ (inherits topology from $M_{\mathbb{Z}}$.) $\widetilde{M} = \{\mu_x \in H_n(M|x) \mid \mu_x \text{ generator}\}$; an orientation in fact gives a section of $\widetilde{M}$.

$$\begin{tikzcd} \widetilde{M} \arrow[dr, "\pi"'] & \subset & M_{\mathbb{Z}} \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

Since we're over $\mathbb{Z}$, each $H_n(M|x)$ has two generators $\Rightarrow$ $\widetilde{M}$ is a <u>double cover</u> of M..

We call $\widetilde{M}$ the orientation double-cover of M, in light of the above definition & also b/c of:

<u>Lem</u>: $\widetilde{M}$ always admits a canonical orientation (even if M doesn't). (note $\widetilde{M}$ is a manifold).

<u>Idea</u>: A point $\tilde{x} \in \widetilde{M}$ is a pair $\tilde{x} = (\mu_x, x)$ where $\mu_x \in H_n(M|x)$ is a generator.

Observe that $H_n(\widetilde{M}|\tilde{x}) \cong H_n(M|x)$ (b/c $\widetilde{M}$ covers M sending $\tilde{x}$ to x); so orient by, at $\tilde{x} = (\mu_x, x)$,

choosing the generator $\mu_x \in H_n(\widetilde{M}|\tilde{x}) \cong H_n(M|x)$. ⊠. ↙ exercise: fill in details / check continuous.

On the other hand, M itself may not be <u>orientable</u> (meaning admit an orientation).

<u>Prop</u>: Say M connected. Then M is orientable $\Leftrightarrow$ $\widetilde{M}$ has two connected components.

<u>Pf</u>: $\widetilde{M}$ is a 2-sheeted cover, hence only has 1 or 2 components.

<u>If 2 components</u>: each maps homeomorphically to M, so M is orientable (pick a section by picking one component of $\widetilde{M}$ & mapping M to that component by inverse of covering identification)

<u>If M orientable</u>: It has exactly two orientations since it's connected.

(point: given an orientation $\{\mu_x\}_{x\in M}$: μ_x determines μ_y at any point in same component as M by this picture:)

So all we can do is swap $\mu_x \mapsto -\mu_x$; this forces '$\{\mu_x\}_{x\in M} \rightsquigarrow \{-\mu_x\}_{x\in M}$.).

⇒ ∃ exactly two sections $s_1, s_2 : M \longrightarrow \tilde{M}$ ~~there~~ w/ disjoint images.

Each gives a component of $\tilde{M}$ (point is that given any section $M \xrightarrow{s} \tilde{M}$ of a covering space $s(M)$ is an entire component of $\tilde{M}$ — why? (exercise: show open + closed)).

$\blacksquare$.

R-case: A generator in $H_n(M|x;R) \cong R$ is a unit/invertible element.

(sometimes more than 2 elts, sometimes fewer! e.g., $R = \mathbb{Z}/2$)

Note: $H_n(M|x;R) \cong H_n(M|x;\mathbb{Z}) \otimes_{\mathbb{Z}} R$ (by UCT for homology — why?), b/c $H_{n-1}(M|x;\mathbb{Z})$ is zero $(n>1)$ or free $(n=1)$).

so each $r \in R$ determines a subcovering space M_r of M_R consisting of all elements of the form $\pm \mu_x \otimes r \in H_n(M|x;R)$, μ_x any generator in $H_n(M|x)$.

- If r is a 2-torsion element (including the case $r=0$), then $r=-r$, so M_r is a copy of M. (i.e., $M_r \xrightarrow{\cong} M$).
- Otherwise $M_r \cong \tilde{M} \cong M_{-r}$, and $M_R = \coprod_{\{r,-r\} \in R/\{\pm 1\}} M_r$.

Using this decomposition, we see that:

(1) An orientable manifold is R-orientable for all R.

(2) A non-orientable manifold is still R-orientable if R contains a unit of order 2. (e.g., if $2=0$ in R).

In particular, every manifold is $\mathbb{Z}/2$-orientable. (point: there's always a section of $M_{1 \in \mathbb{Z}/2} \subset M_{\mathbb{Z}/2}$ b/c $M_{1 \in \mathbb{Z}/2} \cong M$).

Most important cases: $R = \mathbb{Z}, \mathbb{Z}/2$.

Main theorem: M^n connected manifold, R as before

can think of as $H_n(M|M;R)$

(a) If M is compact and R-orientable, then $H_n(M;R) \longrightarrow H_n(M|x;R) \cong R$ is an isomorphism for every $x \in M$.

(b) If M is compact & non-R-orientable, then $H_n(M;R) \longrightarrow H_n(M|x;R) \cong R$ is injective with image $2\text{-Tors}(R) = \{r \in R \mid 2r = 0\}$ for all $x \in M$.

(c) If M is non-compact, then $H_n(M;R) = 0$.

(d) $H_i(M;R) = 0$ for $i > n$.

In particular:

- For a cpct connected manifol M^n, $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ or 0 depending on whether M is orientable.

(ex: $H_2(\mathbb{RP}^2;\mathbb{Z}) \underset{\text{by CW homology}}{=\!=} 0$ so $\mathbb{RP}^2$ not orientable. $H_3(\mathbb{RP}^3;\mathbb{Z}) = \mathbb{Z}$ so $\mathbb{RP}^3$ is orientable).

- either way if M^n cpct, connected, $H_n(M;\mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z}$.

Def'n: M orientable and compact. An element of $H_n(M;R)$ whose image in $H_n(M|x;R)$ generates for all x is called a fundamental class for M with R-coeffs, denoted $[M]$. (note this is a choice).

A fund. class $[M] \in H_n(M;\mathbb{Z})$ is a generator, and is equivalent, for a cpct. manifold, to a choice of orientation (as we'll see).

Cor: A fund. class $[M]$ w/ R-coeffs. exists iff M is cpct. and R-orientable.

($\Leftarrow$ Thm, $\Rightarrow$ Say $[M]$ is a fund. class; since $[M] \neq 0$, M cpct, let μ_x be its image in $H_n(M|x;R)$.
observe (exercise): $x \longmapsto (\mu_x, x)$ is an orientation of M, i.e., is continuous.)

More technical statement (than theorem), implies main theorem:

M^n connected manifold (not. nec. cpct.), A closed subset of M, and given $M_R \xrightarrow{\pi} M$, consider $(M_R)|_A := \pi^{-1}(A)$; have $(M_R)|_A \xrightarrow{\pi|_A} A$, & denote its sections by $\Gamma(A;(M_R)|_A)$.

Lem: (a) There is a bijection, for A $\underline{\text{compact}}$:

$$\Gamma(A; (M_R)|_A) \xleftarrow[J_A]{1:1} H_n(M|A;R)$$

$$s_{\alpha_A} \longleftarrow\!\!\shortmid \overset{\in}{\alpha_A}$$

defined by

$$s_{\alpha_A} : x \longmapsto \underbrace{(\alpha_A)|_x}$$ denotes image of α_A under $H_n(M|A;R) \to H_n(M|x;R)$

(For A not necessarily compact — we won't prove this case —

$$(\star) \qquad \Gamma_c(A; (M_R)|_A) \xleftarrow[J_A]{\cong} H_n(M|A;R)$$)

Γ_c: sections w/ cpct. support, meaning $s_x \equiv 0$ for x outside a compact set in base.

J_A (defined as above).

(b) $H_i(M|A;R) = 0$ for $i > n$, A closed.

Claim: Lem $\Rightarrow$ Main Theorem. Assume lemma.

(Need $\star$ to show Main thm if M non-cpct; we will omit this).

If M cpct, then $A = M$ is compact.

- part (b) of lemma w/ $A = M$ implies $H_i(M|M;R) = H_i(M;R) = 0$ $i > n$.
- part (a) of lemma w/ $A = M$ cpct implies

$$(\star) \qquad \Gamma(M_R) \xleftarrow[J_M]{\cong} H_n(M;R) \quad \text{as } R\text{-modules.}$$

$$\{s_\alpha : x \mapsto \alpha|_x\} \longleftarrow\!\!\shortmid \alpha$$

Now M is connected, so a section is determined by its value at a point (b/c M_R covering space)

We'll use $(\star)$ to study the relationship between $H_n(M;R)$ and orientations, which are after all certain sections of M_R.

——— 2/17 ———

Recall: M^n manifold

Defined 'local homology groups' $H_n(M|x;R) := H_n(M, M-x; R)$ for $x \in M$

and $H_n(M|A;R) := H_n(M, M-A; R)$. for $A \subseteq M$

& know $H_n(M|x;R) \cong R$ & $H_n(M|A;R) \xrightarrow{\cong} H_n(M|x;R)$

if A convex subset of a Euclidean chart, and $x \in A$.

More generally, we always have a restriction map

$H_n(M|B;R) \to H_n(M|A;R)$ for $A \subseteq B$, but may not always be an iso.

We constructed a 'bundle of R-modules' $M_R \xrightarrow{\pi} M$ whose fiber at $x \in M$ was $H_n(M|x;R)$. (covering space).

orientation double cover

(2:1)

If $\widetilde{M} \subset M_{\mathbb{Z}}$ is the subcovering space whose fibers are generators of $H_n(M|x;\mathbb{Z})$,

we showed $M_R = \bigcup_{(r,-r)\in R} M_r$, where $M_r = \begin{cases} M & r \text{ 2-torsion} \\ \widetilde{M} & \text{otherwise.} \end{cases}$

fiber at x is everything of the form $\pm \mu_x \otimes r$

generator of $H_n(M|x;\mathbb{Z})$.

M orientable if $\widetilde{M} := M \sqcup M$

$\iff$ if $\widetilde{M} \xrightarrow{\pi} M$ admits a section. (M R-orientable if $\exists$ a section of M_R which generates each fiber).

From last time

Main theorem: M^n connected manifold, R as before

can think of as $H_n(M|M;R)$

(a) If M is compact and R-orientable, then $H_n(M;R) \longrightarrow H_n(M|x;R) \cong R$ is an isomorphism for every $x \in M$. (converse is more easily true).

(b) If M is compact & non-R-orientable, then $H_n(M;R) \longrightarrow H_n(M|x;R) \cong R$ is injective with image $2\text{-Tors}(R) = \{r \in R \mid 2r = 0\}$ for all $x \in M$.

(c) If M is non-compact, then $H_n(M;R) = 0$.

(d) $H_i(M;R) = 0$ for $i > n$.

Lem: (a) There is a bijection, for $A \overset{\subsetneq M}{\underline{\text{compact}}}$:

$$\begin{tikzcd} \Gamma(A;(M_R)|_A) & H_n(M|A;R) \arrow[l, "1:1", "J_A"'] \\ s_{\alpha_A} & \alpha_A \arrow[u, phantom, sloped, "\in"] \arrow[l, mapsto] \end{tikzcd}$$

defined by

$s_{\alpha_A}: x \longmapsto \underline{(\alpha_A)}|_x$ ← denotes image of α_A under $H_n(M|A;R) \to H_n(M|x;R)$

(For A not necessarily compact — we won't prove this case —

$$(\#) \quad \Gamma_c(A;(M_R)|_A) \xleftarrow[J_A]{\cong} H_n(M|A;R) \quad)$$

sections w/ cpct. support, meaning $s_x \equiv 0$ for x outside a compact

(defined as above).

(from before)

(b) $H_i(M|A;R) = 0$ for $i > n$, A closed.

Claim: Lem ⇒ Main Theorem. Assume lemma, for all A.

- part (b) implies ($A=M$) $H_i(M;R)=0$ $i>n$.
- if M is non-compact, observe that $\Gamma_c(M;M_R) = 0$, because a section of a covering space (if it exists) is determined on any connected component by what it does at a single point (covering space theory)

(M cpct, set $A=M$)

- It suffices by lemma to study

$$\begin{tikzcd} H_n(M;R) \arrow[r, phantom, "\cong"] \arrow[rr, bend left, "\text{restr.}"] & \Gamma(M;M_R) \arrow[r, "\text{restr.}"] & (M_R)_x := H_n(M|x;R) \cong R. \\ & s \arrow[r, mapsto] & s_x. \end{tikzcd}$$

for any $x \in M$.

* restr is always injective, b/c any section if it exists is determined by its values at a point (M is connected), by covering space theory.

- If M is orientable, and $r \in R$ then there is a section of M_R taking value r at $(M_R)_x$, b/c $M_r \cong \left\{ \tilde{M} \right.$ and we can find a section of M over both $\tilde{M}$ and M;

$\subseteq M$

hence over $M_r \subseteq M_R$.

- If M is not orientable, then we can only find a section of M_r when ~~[illegible]~~ r is 2-torsion, hence the image of restr consists of 2-torsion. $\boxtimes$.

<u>Pf of technical lemma</u>: (sketch, in the case A is compact) :

omit R from notation for this proof, for simplicity.

The idea is to induct on the size of A and M.

Let <u>$P_M(A)$</u> be the statement that $J_A : H_n(M|A) \xrightarrow{\cong} \Gamma(A; (M_R)|_A)$ is an iso.

<u>Claim 1</u>: If $P_M(A)$, $P_M(B)$, and $P_M(A \cap B)$ hold, then $P_M(A \cup B)$ holds.

<u>Idea</u>: First observe for G abelian, $H_1, H_2 < G$, there's a SES

$$(g_1+H_1,\, g_2+H_2) \longmapsto (g_1-g_2+H_1+H_2).$$

$$0 \to \frac{G}{H_1\cap H_2} \to \frac{G}{H_1} \oplus \frac{G}{H_2} \to \frac{G}{H_1+H_2} \to 0,$$

$$g+(H_1 \cap H_2) \longmapsto (g+H_1,\, g+H_2)$$

For $V_1, V_2 \underset{\text{open}}{\subset} X$ recall we defined "$C_\bullet(V_1+V_2)$" $:=$ sum $\overset{\text{"}H_1\text{"}}{\downarrow}C_\bullet(V_1) + \overset{\text{"}H_2\text{"}}{\downarrow}C_\bullet(V_2)$

(in $C_\bullet(X) \leftarrow$ "G"

$\Rightarrow$ a SES $\quad 0 \to C_\bullet(X, V_1 \cap V_2) \to C_\bullet(X, V_1) \oplus C_\bullet(X, V_2) \to C_\bullet(X, V_1+V_2) \to 0$

$\uparrow$ know $H_\bullet$ of this computes $H_\bullet(X, V_1 \cup V_2)$ from before

$\boxed{\text{LES}}$
$\Rightarrow$
("M-V upside down")

$$\cdots \to H_{n+1}(X, V_1\cup V_2) \to H_n(X, V_1 \cap V_2) \to H_n(X, V_1) \oplus H_n(X, V_2) \to H_n(X, V_1 \cap V_2) \to \cdots$$

our <u>case</u>: study the case
- $X = M$
- $V_1 = M - A$
- $V_2 = M - B$

- $V_1 \cap V_2 = M - (A \cup B)$
- $V_1 \cup V_2 = M - (A \cap B)$.

$H_i(M, V_1), H_i(M, V_2)$

$H_i\ (M, V_1 \cup V_2) = 0$, $\overset{\text{when}}{i > n}$ by assumption. So M-V implies immediately that

$H_i\ (M, V_1 \cap V_2) = 0$ for $i > n$ too (sandwiched between 0's in LES).

We also get a diagram of SES's :

$$\begin{tikzcd}
H_{n+1}(M, V_1 \cup V_2) \arrow[d, equal] & & & \\
0 \arrow[r] & H_n(M|A\cup B) \arrow[r, "(\mathrm{restr}_A, \mathrm{restr}_B)"] \arrow[d, "J_{A\cup B}"] & H_n(M|A) \oplus H_n(M|B) \arrow[r, "(\text{difference of restrictions})"] \arrow[d, "J_A \oplus J_B", "\cong"'] & H_n(M|A\cap B) \arrow[d, "J_{A\cap B}", "\cong"'] \\
0 \arrow[r] & \Gamma(A\cup B, M_R) \arrow[r, "(\mathrm{restr}_A, \mathrm{restr}_B)"'] & \Gamma(A; M_R) \oplus \Gamma(B; M_R) \arrow[r, "(\text{difference of restricts})"'] & \Gamma(A\cap B; M_R)
\end{tikzcd}$$

exercise: Check lower SES — a more general fact about sections — & commute diagram

5-Lemma $\Rightarrow$ $J_{A\cup B}$ is an isomorphism.

Now, using Claim 1, we can already reduce to the case of $M = \mathbb{R}^n$, A some cpct. set.

How? If $A \subseteq M$ cpct subset, can write $A = A_1 \cup \cdots \cup A_m$ where each A_i is cpct and contained in an open $\mathbb{R}^n \subset M$. (why? exercise).

Note first of all that if $A_i \subset \mathbb{R}^n \subset M$, then $P_M(A_i) \Longleftrightarrow P_{\mathbb{R}^n}(A_i)$ (b/c by excision $H_n(M|A_i) \cong H_n(\mathbb{R}^n|A_i)$).

Assuming $P_{\mathbb{R}^n}(B)$ holds for any B cpct for a moment, suppose inductively that $P_M(A_1 \cup \cdots \cup A_{m-1})$ and $P_M(A_m)$, The intersection $(A_1 \cap A_m) \cup \cdots \cup (A_{m-1} \cap A_m)$ is again a union of $(m-1)$ cpct subsets of Euclidean charts, so $P_M((A_1 \cup \cdots \cup A_{m-1}) \cap A_m)$ holds too.

for any cpct $A_i \subset$ some $\mathbb{R}^n \subset M$.

Then Claim 1 $\Rightarrow$ $P_M(A_1 \cup \cdots \cup A_m)$ holds.

Claim 2: If $M = \mathbb{R}^n$, A convex subset, then the result is true - because we've already shown

$$\begin{tikzcd}
H_n(M|A) \arrow[r, "\cong"] \arrow[d, "J_A"] & H_n(M|x) \arrow[d, "\mathrm{id} = J_x", "\cong"'] & (x \in A) \\
\Gamma(A, M_R) \arrow[r, "\mathrm{restr.}", "\cong"'] & \Gamma(x; M_R) &
\end{tikzcd}$$

$\Rightarrow J_A \cong$.

(by contractibility.)

exercise

What to do for an arbitrary compact set $A \subset \mathbb{R}^n$? If $A = \bigcup_{\text{finite}}$ convex sets, we're done by Claim 1. (b/c intersection of convex sets is convex).

Idea is that any A can be 'approximated' by unions of convex sets.

in the sense that $\exists$ $E_1, E_2, E_3, \ldots,$ seq. of compact sets in $\mathbb{R}^n$

with $E_1 \supset E_2 \supset E_3 \supset \cdots$, each E_i is a (finite) union of convex sets,

and $\bigcap E_i = A$.

(e.g., pick $\delta_1, \delta_2, \delta_3, \ldots$ $\delta_i \to 0$, let E_1 be any finite cover of A by closed δ_1-balls,

and let E_k be intersection of E_{k-1} w/ any finite cover of A by δ_k-balls

(intersections preserve the property of being a finite union of convex sets)).

How does this help?

Claim 3: If $P_M(A_i)$ holds for $A_1 \supset A_2 \supset A_3 \supset \cdots$ seq. of cpt. subsets then

$P_M(A = \bigcap A_i)$ holds.

(in light of above, it follows $P_{\mathbb{R}^n}(A)$ holds for any cpt. A hence $P_M(A)$ holds for any A).

Sketch of claim:

Have a diagram

$$\begin{tikzcd}
H_n(M|A_1) \arrow[r, "restr."'] \arrow[d, "J_{A_1}"', "\cong"] \arrow[rrrr, bend left, "restr."] \arrow[dr, phantom, "\circlearrowleft"] & H_n(M|A_2) \arrow[r, "restr."'] \arrow[d, "J_{A_2}", "\cong"'] \arrow[rrr, bend left] \arrow[dr, phantom, "\circlearrowleft"] & H_n(M|A_3) \arrow[r] \arrow[d, "J_{A_3}", "\cong"'] \arrow[rr, bend left] & \cdots & H_n(M|A) \arrow[d, "J_A (?)"] \\
\Gamma(A_1; M_R) \arrow[r, "restr."] \arrow[rrrr, bend right] & \Gamma(A_2; M_R) \arrow[r, "restr."] \arrow[rrr, bend right] & \Gamma(A_3; M_3) \arrow[r] \arrow[rr, bend right, "restr."] & \cdots & \Gamma(A; M_R)
\end{tikzcd}$$

This induces a map

$$\begin{tikzcd}
\varinjlim_i H_n(M|A_i) \arrow[r, "restr.", "(\cong \text{ claim 1})"'] \arrow[d, "\varinjlim J_{A_i}", "\cong \text{ by hyp.}"'] & H_n(M|A) \arrow[d, "J_A \ ?"] \\
\varinjlim_i \Gamma(A_i; M_R) \arrow[r, "restr.", "\cong (\text{claim 2})"'] & \Gamma(A; M_R)
\end{tikzcd}$$

general claim 1: $\varinjlim_i H_p(M|A_i) \xrightarrow{\cong} H_p(M|A)$ for any p.

If general claim 1 and claim 2 are both true, then $P_M(A)$ holds.

where $\varinjlim$ denotes the direct limit.

Recap: The direct limit of $G_1 \to G_2 \to G_3 \to \cdots$ satisfies univ. property

$$\begin{tikzcd}
 & & H \\
 & & \varinjlim G_i \arrow[u, dashed, "\exists !"'] \\
G_1 \arrow[r] \arrow[urr, bend left] \arrow[uurr, bend left] & G_2 \arrow[r] \arrow[ur] \arrow[uur] & \cdots \arrow[u] \arrow[uu, bend right]
\end{tikzcd}$$

More directly: $(S, \leq)$ directed set means for all

$\alpha, \beta \in S \quad \exists\, \gamma$ with $\alpha \le \gamma, \beta \le \gamma$ (e.g., $\mathbb{N}, \le$).

Given $(G_\alpha, \alpha \in S)$ and $\psi_{\alpha\beta} : G_\alpha \to G_\beta$ when $\alpha \le \beta$

with $\psi_{\beta\gamma}\psi_{\alpha\beta} = \psi_{\alpha\gamma}$ if $\alpha \le \beta \le \gamma$,

then $\varinjlim G_\alpha := G := \{(g,\alpha) \mid g \in G_\alpha\} / (g,\alpha) \sim (h,\beta)$

if $\exists\, \gamma$ with $\alpha \le \gamma, \beta \le \gamma$,

and $\psi_{\alpha\gamma}(g) = \psi_{\beta\gamma}(h)$ in G_γ.

We'll mostly leave general claim 1 & claim 2 to be exercises, but we want to indicate one key idea (for general claim 1).

why is

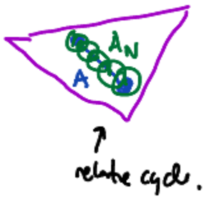
$$\begin{tikzcd}
\varinjlim_i H_p(M|A_i) \arrow[r] \arrow[d, equal] & H_p(M|A) \arrow[d, equal] \\
\varinjlim_i H_p(M, M-A_i) \arrow[r] & H_p(M, M-A) \\
 & {[\sigma]} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

surjective?

The first observation is that any relative cycle σ in $(M, M-A)$ has $\partial\sigma$ compact, hence supported in a compact subset of $M-A$. This implies its disjoint from some A_N, $N >> 0$, hence contained in $(M, M-A_N)$.

(why? exercise:

idea: (in $\mathbb{R}^n$)



$\partial\sigma$ and A have a minimum distance δ, hence $\partial\sigma$ doesn't touch any cover of A by closed $\delta/2$ balls either).

Claim 2 follows eventually from $(M-A) = \bigcup (M-A_i)$. ☒

2/22/2021

Poincaré duality:

(Right now $R = \mathbb{Z}$
implicitly, can work w/ R-coeffs, then M R-orientable; i.e., M always $\mathbb{Z}/2$-orientable)

Thm says (first version):

If M^n orientable, cpt manifold, then $H^\ell(M) \cong H_{\overset{\text{dimension }(M).}{n}-\ell}(M)$

(M R-orientable, $H^\ell(M;R) \xrightarrow{\cong} H_{n-\ell}(M;R)$).

The isomorphism is given by cap product with a fundamental class:

recall that we have cap product action $H_n(X) \times H^\ell(X) \longrightarrow H_{n-\ell}(X)$,

assume M connected

and if M R-orientable, a fund. class is a choice of generator $[M] \in H_n(M;R) \cong R$

$\Longleftrightarrow$ a choice of section of $M_R \to M$ which generates at each fiber, i.e., an R-orientation.

$$\rightsquigarrow D_M := [M] \frown (-) : H^\ell(M;R) \longrightarrow H_{n-\ell}(M;R)$$

duality isomorphism.

---

originally historically phrased in terms of existence of a dual polyhedral subdivision to a given sufficiently fine triangulation.

point $\longleftrightarrow$ top-dim'l face

1-simplex $\longleftrightarrow$ codim-1 face.

$\vdots \qquad \vdots$

compatible in a dual sense w/ boundary operators.

Corollaries of Poincaré duality

M oriented, n'dim'l, cpct.

knew this

In principle could have had $\text{Ext}(H_{n-1}, \mathbb{Z})$ contributions.

(1) If M connected, then $H_n(M) = \mathbb{Z}$, and $H^n(M) = \mathbb{Z}$ (b/c $H^0(M) = \mathbb{Z}$ and $H_0(M) = \mathbb{Z}$).

(2) Let's use the notation

$$\overline{H} := H / \text{Tors}(H), \text{ for a } \mathbb{Z}\text{-module } H.$$

Poincaré duality implies there's a perfect pairing on $\overline{H}^\bullet(X)$ resp. $\overline{H}_\bullet(X)$.

(Recall if $\Gamma_1 \cong \mathbb{Z}^n$, $\Gamma_2 \cong \mathbb{Z}^n$, a bilinear $q : \Gamma_1 \times \Gamma_2 \longrightarrow \mathbb{Z}$ is perfect iff

$$q^* : \Gamma_1 \xrightarrow{\cong} \text{Hom}(\Gamma_2, \mathbb{Z})$$
$$e \longmapsto q(e,-)$$

$\Longleftrightarrow$ for any $\mathbb{Z}$-bases of Γ_1, Γ_2, matrix of q has det ± 1. (unimodular)

To spell out the details, let's recall first that

Thm: M cpct m'fold. Then $H_\ell(M)$ is a finitely generated $\mathbb{Z}$-module for all ℓ.

(we'll omit details, see Hatcher).

appeals to classification of f.g. $\mathbb{Z}$-mods

Using this, we learn $H_\ell(M) = \mathbb{Z}^r \oplus \text{Torsion}$ & $\text{Ext}(H_\ell(M), \mathbb{Z}) \cong \text{Tors}(H_\ell(M))$.

UCT tells us that $H^\ell(M) \to \mathrm{Hom}(H_\ell(M), \mathbb{Z})$ is surjective w/ kernel the torsion of $H_{\ell-1}(M)$.
$[\phi] \longmapsto \beta([\phi])([\sigma]) := \phi(\sigma)$.

" torsion of $H^\ell(M)$

$\Rightarrow$ get $\overline{H^\ell}(M) \xrightarrow[\beta]{\cong} \mathrm{Hom}(\overline{H_\ell(M)}, \mathbb{Z})$.

means mod torsion

by this fact.

$\cong \mathrm{Hom}(H_\ell(M), \mathbb{Z})$

(b/c $\mathrm{Hom}(H, \mathbb{Z})$ kills $\mathrm{tors}(H)$).

i.e., Have a perfect pairing $\overline{H^\ell(M)} \times \overline{H_\ell(M)} \to \mathbb{Z}$

$\langle [\phi], [\sigma] \rangle = \phi(\sigma)$.

P.D. $\Rightarrow \exists$ a perfect pairing

$\overline{H_{n-\ell}(M)} \times \overline{H_\ell(M)} \to \mathbb{Z}$.

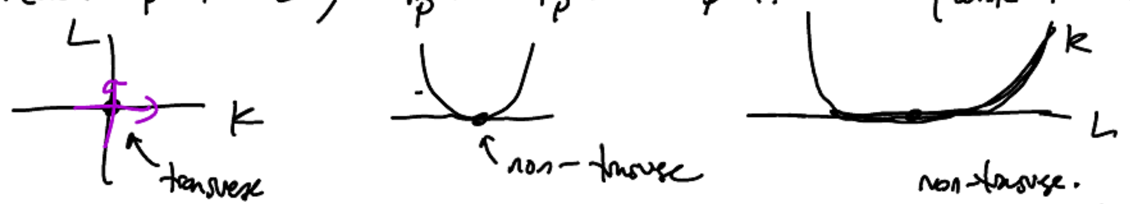
$(\gamma_1, \gamma_2) \longmapsto \gamma_1 \bullet \gamma_2 := \langle D_M^{-1} \gamma_1, \gamma_2 \rangle$

$\overline{H^\ell}(M) \ni D_M^{-1}\gamma_1$

"intersection pairing" (why?)

Geometrically, if $K^{n-\ell}, L^\ell \subseteq M^n$ compact oriented submanifolds of M (cpct oriented)

Let's assume further K, L, M smooth, and K, L intersect transversely, meaning at each $p \in K \cap L$, $T_pK + T_pL = T_pM$. (write $K \pitchfork L$)



(points carry signs: do the sums of orientations on K, L match orientation on M at p?)

When K, L transverse, $K \cap L$ is a cpct oriented 0-manifold (= finite union of points),

$\Rightarrow K \bullet L := \sum_{p \in K \cap L} \mathrm{sign}(p)$.

geom. intersection #

± 1 depending on

an isotopy of a submanifold $K \overset{i}{\hookrightarrow} M$ is a smooth homotopy i_t, with each i_t an embedding.

If $K \not\pitchfork L$, we can isotope it to be $\pitchfork$ & then intersect.

Intersection # is an isotopy invariant so result is invariant.

defined using P.D.

__Thm__ (omitted here): For K, L as above, $K \bullet_{\text{geom}} L = [K] \bullet [L]$ in $H_\ell(M)$.

means look at image $[K]$ in $H_{n-\ell}(K) \to H_{n-\ell}(M)$.

Duality in terms of cap product.

Thm: (coh. intersection pairing) M^n cpt, (R-)oriented, R implicit. Then, the pairing

$$\begin{tikzcd}
\overline{H}^p(M) \otimes \overline{H}^{n-p}(M) \arrow[r, "\cup"] & \overline{H}^n(M) \arrow[d, "\cong"', "(-)\cap[M]\ (\text{or } D_M)"] \arrow[dd, dashed, bend right] \\
& \overline{H}_0(M) \arrow[d, "\cong"] \\
& R.
\end{tikzcd}$$

is a perfect pairing.

(mod torsion)

sometimes called "$\int_M(-)$."

Recall: if $[\alpha]\in H^\ell(X)$, $[\beta]\in H_\ell(X)$, then $\alpha(\beta) := \varepsilon_*([\alpha]\cap[\beta])$, where $[\alpha]\cap[\beta]\in H_0(X)$, and $\varepsilon_*: H_0(X) \xrightarrow{\cong} R$ (for X connected).

Pf (from P.D.)

$$[\phi] \longmapsto \{[\sigma]\mapsto \phi(\sigma)\}$$

$$\begin{tikzcd}
\text{Have}\quad \overline{H}^p(X) \arrow[r, "\cong", "UCT"'] \arrow[rr, dashed, bend right] & \mathrm{Hom}(\overline{H}_p(X), R) \arrow[r, "\cong", "D_M^*\ (-\circ D_M)"'] & \mathrm{Hom}(\overline{H}^{n-p}(X), R)
\end{tikzcd}$$

This map is given by

$$[\phi] \longmapsto \{[\psi] \longmapsto \phi(\overbrace{[\psi]\cap[M]}^{D_M([\psi])})\}$$

$$= \varepsilon_*([\phi]\cap([\psi]\cap[M]))$$

module property

$$= \varepsilon_*(([\phi]\cup[\psi])\cap[M])$$

$$= (\phi\cup\psi)([M]). \ \blacksquare.$$

chain level version of fund. class

Application: coh. rings of projective spaces

Prop:

$$H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\alpha^{n+1} \quad |\alpha|=1$$

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1} \quad |\alpha|=2 \qquad \text{as rings.}$$

$$H^*(\mathbb{H}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1} \quad |\alpha|=4$$

Pf: Let's do $\mathbb{C}P^n$ (other proofs are the same). Induction on n:

$n=1$: $H^*(\mathbb{C}P^1;\mathbb{Z}) \cong H^*(S^2;\mathbb{Z}) \underset{\text{already know}}{\cong} \mathbb{Z}[\alpha]/\alpha^2 \quad |\alpha|=2.$ ✓.

Inductive step: assume true for $\mathbb{C}P^{n-1}$. $(n>1)$

$\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$ cell, so

LES of $(\mathbb{C}P^n, \mathbb{C}P^{n-1})$ in cohomology $\Rightarrow$ the restriction

$r^*: H^i(\mathbb{C}P^n) \xrightarrow{\cong} H^i(\mathbb{C}P^{n-1})$ for $i \leq 2n-2$. (where $r: \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$)
(why? exercise)

By naturality of cup product, we learn that if $\alpha \in H^2(\mathbb{C}P^n)$ generator, then $r^*\alpha$ generates $H^2(\mathbb{C}P^{n-1})$,

$$\Rightarrow \underset{\substack{\| \text{ naturality} \\ r^*(\alpha^i)}}{(r^*\alpha)^i} \text{ generates } H^{2i}(\mathbb{C}P^{n-1}) \quad i \leq 2n-2 \quad \text{(by induction step).}$$

$\Rightarrow \alpha^i$ generates $H^i(\mathbb{C}P^n)$ $i \leq 2n-2$.

So have elements $\alpha, \alpha^2, \alpha^3, \text{---}, \alpha^{n-1}$ generating $H^2, H^4, \ldots, H^{2n-2}$.

Q: is $\underbrace{\alpha \cup \alpha^{n-1}}_{\alpha^n}$ a generator of $H^{2n}(\mathbb{C}P^n)$? (if so, we're done)

Yes, by Poincaré duality: $\mathbb{C}P^n$ is a cpct $2n$ manifold (connected), & $H_{2n}(\mathbb{C}P^n) \cong \mathbb{Z}$, so orientable. So $\exists$ perfect pairing (choosing $[\mathbb{C}P^n]$):

$$\begin{tikzcd} H^2(\mathbb{C}P^n) \otimes H^{2n-2}(\mathbb{C}P^n) \arrow[r, "\cup"] & H^{2n}(\mathbb{C}P^n) \arrow[r, "D_M"] & H_0(\mathbb{C}P^n) \arrow[d, "\cong"] \\ & & \mathbb{Z}. \end{tikzcd}$$

($\Rightarrow$ a generator $\cup$ a generator must be a generator.). ■.

Ideas in proof of P.D.:

Again by induction/covering argument, want to reduce to case of $\mathbb{R}^n$.

The local case $\mathbb{R}^n$ is a non-compact manifold, for which P.D. as stated fails (e.g., $H_n(\mathbb{R}^n) = 0$ $n>0$).

We need a formulation of P.D. which holds in non-compact setting too, which is suitably functorial allows for induction. We'll get this by replacing $H^\ell \rightsquigarrow H^\ell_c$ "compactly supported cohomology".

(other choice is in Bredon's book).

Say M non-compact mfold, $K \subset M$ cpct. subset. R coef (suppressed).

Recall the cap product for $(M, M-K)$:

$$\underline{H_n(M, M-K)} \times H^\ell(M, M-K) \rightarrow H_{-\ell}(M).$$

$$\|$$

$$\begin{tikzcd} H_n(M|K). \arrow[d, "J_K"', "\cong \ \text{technical lemma (K cpct.)}"] \\ \Gamma(K;M_R). \end{tikzcd}$$

orientations

$\Rightarrow s|_K$

- If M orientable, pick an orientation $(s : M \longrightarrow M_R$ whose image generates $(M_R)_x$ at every $x)$.
- Restrict to K, $s|_K \in \Gamma(K; M_R)$
- Technical lemma says $\exists!\ \mu_K \in H_n(M|K)$ restricting to $s|_K$. Call it the 'local fundamental class' (note if M non-compact $H_n(M) = 0$).

Naively might hope that

$$-\cap \mu_K : H^\ell(M, M-K) \longrightarrow H_{n-\ell}(M)$$

is an iso. for all M, $K \underset{\text{cpct}}{\subset} M$.

(if true, would imply P.D. when M cpct b/c set $K = M$)

This is not exactly true, but it ends up being true in a limiting sense as we let K get arbitrarily large.

Note: If $K_1 \subseteq K_2$ cpct. sets, then

$$(M, M-K_2) \xhookrightarrow[i_{K_2,K_1}]{\text{incl.}} (M, M-K_1),$$

and the element $\overbrace{\mu_{K_2} \text{ maps to } \mu_{K_1}}^{\star}$ (check).

Also get $i^*_{K_2,K_1} : H^\ell(M, M-K_2) \longrightarrow H^\ell(M, M-K_1)$

and if $K_1 \subseteq K_2 \subseteq K_3$ then $i^*_{K_2K_1} \circ i^*_{K_3K_2} = i^*_{K_3K_1}$.

so if we let $S = \{K \mid K \subset M \text{ cpct. subsets}\}$, ordered by $\subseteq$, note S is a directed set

$\{H^\bullet(M, M-K)\}_{K \in S}$ is a system of groups indexed by S using maps $i^*_{L,K}$ for $K \subseteq L$.

Def: The compactly supported cohomology of a (not. nec. compact) manifold M is

$$H^\ell_c(M) := \lim_{\substack{K \subset M \text{ cpct.} \\ (\text{i.e., } K \in S)}} H^\ell(M, M-K)$$

—

(explicitly, H^ℓ_c is given by co-chains $\varphi \in C^\bullet(M)$ w/ $\varphi \equiv 0$ on all chains in $M-K$ for some $K \subseteq M$).

For $K_1 \subseteq K_2$ we claim the following diagram commutes by naturality of cap product (using $\star$) with respect to i_{K_2,K_1} :

$$\begin{tikzcd}
H^\ell(M,M-K_1) \arrow[rd, "\frown \mu_{K_1}"] \arrow[dd, "i^*_{K_2,K_1}"] & \\
\circlearrowright & H_{n-\ell}(M) \\
H^\ell(M,M-K_2) \arrow[ru, "\frown \mu_{K_2}"'] &
\end{tikzcd}$$

(basic property of $\varinjlim_{\alpha\in S} G_\alpha$ (w.r.t. $\varphi_{\alpha\beta} : G_\alpha \to G_\beta$, $\alpha \leq \beta$) is that if have

$\varphi_\alpha : G_\alpha \to H$ w/ $\varphi_\beta \varphi_{\alpha\beta} = \varphi_\alpha$ then get

$\lim \varphi_\alpha = \overline{\varphi} : \varinjlim G_\alpha \to H$

$[(\alpha, g)] \mapsto \varphi_\alpha(g) \ldots$).

So, get a map

$$D_M := \varinjlim_{\substack{K\subset M \\ \text{cpct.}}} (- \frown \mu_K) \; : \; H^\ell_c(M) \longrightarrow H_{n-\ell}(M).$$

Rmk: If M cpct, then $S = \{K \subset M \text{ cpct.}\}$ contains a maximal element, M itself.

By definition of direct limit, can directly verify that if S has a maximal element σ_{max} then

$$G_{\sigma_{max}} \xrightarrow{\cong} \varinjlim_{\alpha\in S} G_\alpha .$$

In this case, we see that

$$\begin{tikzcd}
H^\ell(M,M-M) \arrow[d, phantom, "\shortparallel"] & \\
H^\ell(M) \arrow[r, "\cong"] \arrow[rd, "D_M"'] & H^\ell_c(M) \arrow[d, "D_M"] \\
& H_{n-\ell}(M).
\end{tikzcd}$$

&

Thm: (Poincaré duality for non-compact manifolds):

If M is oriented, then

$$D_M := \varinjlim_{\substack{K\subset M \\ \text{cpct.}}} (- \frown \mu_K) \; : \; H^\ell_c(M) \xrightarrow{\cong} H_{n-\ell}(M).$$

↑ induced by choice of orientation.

(Rmk above says this recovers P.D. for compact manifolds)

Idea of proof:

Induct on M. Let $P(M)$ be the statement above for a given M.

Step 1: True when $M = \mathbb{R}^n$ (hence true when $M = \text{ball in } \mathbb{R}^n$)

Step 2: If $M = U \cup V$, U, V open, & $P(U)$, $P(V)$, $P(U \cap V)$ hold, then $P(U \cup V) = P(M)$ holds.

(w/ step 1 ⇒ true for any finite union of $\overset{\text{open}}{\wedge}$ balls in $\mathbb{R}^n$).

Step 3: (limits): If $P(-)$ holds for each of $\underset{\text{open}}{U_1 \subset} U_2 \underset{\text{open}}{\subset} U_3 \subset \cdots$ (all $\overset{\text{open}}{\text{in}}$ some M) then $P(\bigcup U_i)$ holds.

⇒ $P(-)$ holds for any open in $\mathbb{R}^n$ (can always express any $U \overset{\text{open}}{\text{in}} \mathbb{R}^n$ as union of countably many open balls, & let U_k be union of first k balls. By Step 2 $P(U_k)$ holds & $U_1 \subset U_2 \subset \cdots \underset{\text{step 3}}{\Rightarrow} P(U := \bigcup U_i)$ holds)

⇒ $P(-)$ holds for any finite (by Step 2) & the countable (by Step 3) union of open sets in M which are homeomorphic to $\mathbb{R}^n$.

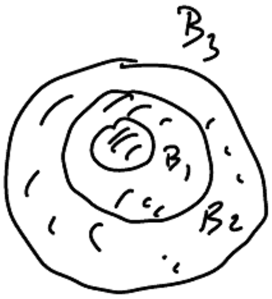
(note given $U_1 \subset M$ $U_1 \cong \mathbb{R}^n$ $U_2 \subseteq M$ $U_2 \cong \mathbb{R}^n$; $U_1 \cap U_2 \cong$ an open in $\mathbb{R}^n$. need step 3 to get $P(U_1 \cap U_2)$, then $P(U_1 \cup U_2)$ by Step 2).

⇒ $P(-)$ holds for M. (we'll assume M has a countable base for simplicity only.)

Step 1: True when $M = \mathbb{R}^n$ (hence true when $M = \text{ball in } \mathbb{R}^n$)

Idea: $\mathbb{R}^n$ is exhausted by cpct. subsets $\overline{B_i(0)}$, $i \in \mathbb{N}$.

⇒ $\overline{B_i(0)}$ is cofinal in $\{K \subset \mathbb{R}^n \text{ compact subsets}\}$



($\underset{\substack{\uparrow \\ \text{direct systems}}}{T} \subseteq \underset{\substack{\nwarrow \\ \text{direct system}}}{S}$ is cofinal in S if every $s \in S$ is $\leq$ some $t \in T$) $\Rightarrow \varinjlim_{s \in S} G_s \xleftarrow{\cong} \varinjlim_{t \in T} G_t$.)

$$\Rightarrow H^l_c(\mathbb{R}^n) \cong \varinjlim_{i} H^l(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B_i}(0))$$

general properties of direct limit

$$\begin{aligned} &\Vert 2 \text{ excision} \\ &H^l(D^n, D^n \setminus \overline{B}_i(0)) \quad D^n \text{ very large disk (maybe depends on } i) \\ &\Vert 2 \text{ htpy inv.} \end{aligned}$$

$$H^\ell(D^n, S^{n-1})$$

$\cong$ good pair.

$$\tilde{H}^\ell(D^n/S^{n-1}) \cong \tilde{H}^\ell(S^n) \cong \begin{cases} \mathbb{Z} & \ell = n \\ 0 & \text{otherwise.} \end{cases}$$

Study:

$$H^\ell(\mathbb{R}^n, \mathbb{R}^n \setminus B_i(0)) \xrightarrow{\frown \mu_{\overline{B_i(0)}}} H_{n-\ell}(\mathbb{R}^n).$$

- $\cong$ when $\ell \neq n$ b/c both sides are 0.
- when $\ell = n$, $\cong$ b/c $\mu_{\overline{B_i(0)}}$ is a generator of $H_n(\mathbb{R}^n | \overline{B_i(0)})$,

and UCT says that

$$\begin{tikzcd}
H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_i(0)) \arrow[r, "\cong"] \arrow[dr] & \mathrm{Hom}_{\mathbb{Z}}(H_n(\mathbb{R}^n | \overline{B_i(0)}), \mathbb{Z}) \arrow[d, phantom, "\cong"] \\
{[\phi]} \arrow[dr, mapsto] & \mathbb{Z} \\
& \phi(\mu_{\overline{B_i(0)}}) \arrow[d, phantom, "="] \\
& \varepsilon_*([\phi] \frown [\mu_{\overline{B_i(0)}}])
\end{tikzcd}$$

Hence $- \frown [\mu_{B_i(0)}]$ is an isomorphism.

Now, provided we know that

$$\begin{tikzcd}
H^\ell(\mathbb{R}^n, \mathbb{R}^n \setminus B_i(0)) \arrow[r, "\cong"] \arrow[dr, "- \frown [\mu_{B_i(0)}]"', "\cong"] & H^\ell(\mathbb{R}^n, \mathbb{R}^n \setminus B_j(0)) \arrow[d, "- \frown [\mu_{B_j(0)}]", "\cong"'] \\
& H_{n-\ell}(\mathbb{R}^n)
\end{tikzcd}$$

for $j > i$,

we're done.

(exercise)

∎