Last time:

Thm: (Poincaré duality for non-compact manifolds): (R implicit)

If M is oriented, not necessarily compact top. manifold

$$(*) \quad D_M := \varinjlim_{\substack{K\subset M\\ \text{cpct.}}} (- \frown \mu_K) \; : \; H^{\ell}_c(M) \xrightarrow{\cong} H_{n-\ell}(M).$$

induced by choice of orientation.

compactly supported cohomology:

$$H^{\ell}_c(M) := \varinjlim_{\substack{K\subset M\\ \text{cpct.}}} H^{\ell}(M, M-K)$$

$$\mu_K := (-)\frown[M] : H^{\ell}(M,M-K) \to H_{n-\ell}(M).$$

Let $P(M)$ be the above assertion $(*)$ for a given M.

We had reduced the proof of this theorem inductively to establishing 3 assertions:

(1) (last time): True when $M = \mathbb{R}^n$.

(2) If $M = U\cup V$, U, V open, & $P(U)$, $P(V)$, $P(U\cap V)$ hold, then $P(U\cup V) = P(M)$ holds.

sketch these today.

(3) (limits): If $P(-)$ holds for each of $U_1 \underset{\text{open}}{\subset} U_2 \underset{\text{open}}{\subset} U_3 \subset \cdots$ (all open in some M) then $P(\bigcup U_i)$ holds.

A flavor of how step (2) is proved:

The key claim is if $M = U\cup V$, U, V open in M,

Lem: $\exists$ a commutative diagram of Mayer-Vietoris LES:

a covariantly induced pushforward for $U\subset M$, $V\subset M$!

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}", "(1)"'] & H^k_c(U)\oplus H^k_c(V) \arrow[r] \arrow[d, "D_U \oplus -D_V", "(2)"'] & H^k_c(M) \arrow[r, "\delta^*"] \arrow[d, "D_M", "(3)"'] \arrow[dr, phantom, "\circlearrowleft \text{ is a bit technical. } (4)"] & H^{k+1}_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r, "\partial_*"] & H_{n-k-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

usual M-V LES in homology. (Pf: Hatcher p.246 Lemma 3.36).

Assuming lem, if $P(U)$, $P(V)$, $P(U\cap V)$ hold, then (1), (2), (4) are $\cong$, hence 5-Lemma $\Rightarrow$ (3) $\cong$, so $P(M)$ holds.

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one observation is that H^ℓ_c is in fact covariantly functorial for open inclusions $U_1 \underset{\text{open}}{\subset} U_2$.

i.e., $U_1 \overset{i}{\underset{\text{open}}{\subset}} U_2 \rightsquigarrow i_! : H^\ell_c(U_1) \to H^\ell_c(U_2)$ "extension by zero".

These maps appear in top LES, (w.r.t. $U\cap V \hookrightarrow U$, $U\cap V \hookrightarrow V$, $U,V \hookrightarrow M$.

A flavor of Step(3): The main idea is that $(U_1 \overset{i_1}{\underset{\text{open}}{\subset}} U_2 \overset{i_2}{\underset{\text{open}}{\subset}} \cdots) \subseteq M$

induces

$$\begin{tikzcd}
H^\ell_c(U_1) \arrow[r, "(i_1)_!"] \arrow[dr, "(j_1)_!"'] & H^\ell_c(U_2) \arrow[r, "(i_2)_!"] \arrow[d, "(j_2)_!"] & \cdots \arrow[dl] \\
 & H^\ell_c(\bigcup U_i) &
\end{tikzcd}$$

$i_m : U_m \hookrightarrow U_{m+1}$

$j_m : U_m \hookrightarrow \bigcup U_i$

and also

$$\begin{tikzcd}
H_{n-\ell}(U_1) \arrow[r, "(i_1)_*"] \arrow[dr, "(j_1)_*"'] & H_{n-\ell}(U_2) \arrow[r, "(i_2)_*"] \arrow[d, "(j_2)_*"] & \cdots \\
 & H_{n-\ell}(\bigcup U_i) &
\end{tikzcd}$$

hence:

$$\begin{tikzcd}
\varinjlim H^\ell_c(U_i) \arrow[r, "\varinjlim (j_m)_!"] \arrow[d, "\varinjlim D_{U_i}", "\text{112 (given)}"'] & H^\ell_c(\bigcup U_i) \arrow[d, "D_{\bigcup U_i}\ (\cong ??)"] \\
\varinjlim H_{n-\ell}(U_i) \arrow[r, "\varinjlim (j_m)_*"] & H_{n-\ell}(\bigcup U_i)
\end{tikzcd}$$

Main Claim: These are both $\cong$.

Exercise: verify main claim. (basic idea for homology is e.g., that any $\sigma : \Delta^n \to \bigcup U_i$ has image in some U_N, $N \gg 0$).

There are many generalizations of Poincaré duality, we'll focus on one such for manifolds with boundary ("Lefschetz duality" or "Poincaré-Lefschetz", ...)

Def: An n-manifold with boundary is a Hausdorff space M which is locally homeomorphic to either

$\mathbb{R}^n$ or $\mathbb{H}^n = \{x_1 \geq 0\} \subseteq \mathbb{R}^n$.

equality allowed.

Obs: • If $x \in M$ has a nhood homeo. to $\mathbb{R}^n$, then excision as before implies that $H_n(M|x) \left(\overset{\text{def}}{=} H_n(M, M-x) \right) \cong \mathbb{Z}$.

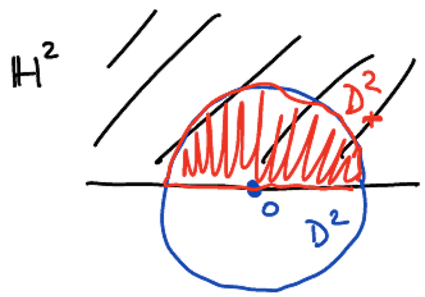
(WLOG $x=0$)

• If $x \in M$ has a nhood homeo. to $\mathbb{H}^n$ in a way sending x to a point with $x_1 = 0$, then excision $\Rightarrow$ $H_n(M|x) \overset{\text{excision}}{\cong} H_n(\mathbb{H}^n, \mathbb{H}^n - \{0\})$

$$\cong \text{excision}$$

$$= H_n(\underbrace{D^n \cap \mathbb{H}^n}_{D^n_+}, D^n_+ \setminus 0)$$

note: both convex sets!
(in contrast, $D^n \setminus 0$ not convex but $D^n_+ \setminus 0$ is)



$$= 0.$$

We conclude that if x is sent to a boundary point in one $\mathbb{H}^n$ local model, it must be sent to a boundary point in every $\mathbb{H}^n$ local model.

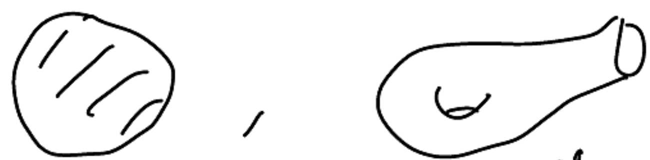
The boundary points of M, denoted ∂M, are those x with $H_n(M|x) = 0$.

($\Leftrightarrow$ those x around which $\exists$ an identification w/ $\mathbb{H}^n$ sending x to the boundary of $\mathbb{H}^n$).

e.g., $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$!

& more generally, $\partial M = (n-1)$-manifold.

examples:



$\partial D^n = S^{n-1}$, $\partial M = S^1$.

A collar neighborhood of ∂M in M is a nhood U of ∂M (in M) homeomorphic to $\partial M \times [0,1)$ (in a way identifying $\partial M \times \{0\} \underset{id}{\cong} \partial M$).

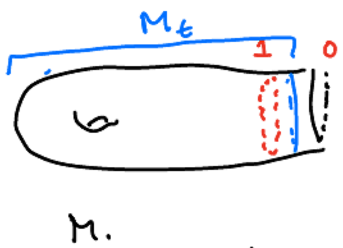
Prop: Any compact manifold with boundary has a collar neighborhood around ∂M.

We'll omit the proof, but note in the smooth case it can be proven by flowing for small time by an inward-pointing vector field

(in turn, an inward-pointing vec. field exists by a partition of unity argument.

General case: Hatcher's book.

Useful consequences of having a collar nhood:



Fix $U \underset{\text{open}}{\supset} \partial M$ with $U \underset{\text{homeo}}{\cong} \partial M \times [0,1)_r$.

We'll define for any $t \in (0,1)$, $M_t := M \setminus (\text{image in } M \text{ of } \{r < t\} \text{ in } U)$.

Observe: $M_t \xhookrightarrow{\text{incl.}} M$ is a homotopy equivalence, and moreover is homotopic to a homeomorphism which is the identity outside the collar, and in the collar, is any homeo

$$[t,1) \xrightarrow{\cong} [0,1) \text{ which is the identity near } 1.$$

(gives $\partial M \times [t,1) \cong \partial M \times [0,1)$, now extend by identity + get $M_t \xrightarrow{\cong} M$).

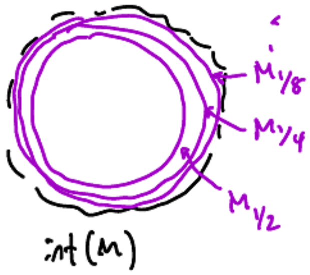
More generally, for $t_1 < t_2$, $M_{t_2} \xhookrightarrow[\text{incl.}]{} M_{t_2}$ is a homotopy equivalence.)

Now look at $\text{int}(M) := M \setminus \partial M$. This is a not necessarily compact manifold (compact iff M cpct & $\partial M = \emptyset$). The above choice of collar nhood & M_t's give us an exhaustion of $\text{int}(M)$ by compact sets

$$M_{s_1} \subset M_{s_2} \subset M_{s_3} \subset \cdots$$

where $s_1 > s_2 > s_3 > \cdots$ is a sequence tending to zero.

e.g.,



Also, each $M_{s_i} \hookrightarrow \text{int}(M)$ is a homotopy equivalence.

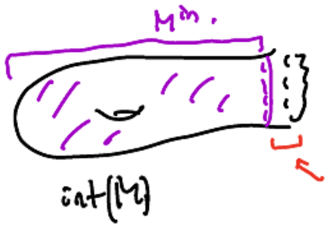
Orientations on manifolds with boundary:

M manifold with boundary. Say M is orientable if $\mathrm{int}(M)$ is orientable, & an orientation on M means an orientation on $\mathrm{int}(M)$.

As before, we can define $\begin{array}{c} M_R \\ \downarrow \\ \mathrm{int}(M) \end{array}$ covering space ('bundle of R-modules') whose fiber at $x \in \mathrm{int}(M)$ is $H_n(M|x; R)$.

Sections of M_R which generate at each point $\longleftrightarrow$ R-orientations

Pick M^{in} m'fold with boundary inside $\mathrm{int}(M)$ homotopy equivalent to $\mathrm{int}(M)$, i.e., $M^{in} := M_{1/2}$.



$\mathrm{int}(M) \setminus M^{in} = \partial M \times$ open interval

M^{in} is compact, inside $\mathrm{int}(M)$, so

Technical lemma w/ $K = M^{in} \hookrightarrow \mathrm{Int}(M)$ implies:

$$\begin{array}{ccccc} \Gamma(M^{in}; M_R) & \cong & H_n(\mathrm{int}(M) | M^{in}) & = & H_n(\underbrace{\mathrm{int}(M)}, \underbrace{\mathrm{int}(M)\setminus M^{in}}) \\ \| \wr & & \uparrow \wr & & \\ \Gamma(\mathrm{int}(M); M_R) & & H_n(M_{1/4}, \partial M_{1/4}) & & \\ & & \| \wr & & \\ & & H_n(M, \partial M). & & \end{array}$$

$M^{in} \simeq \mathrm{int}(M)$ h.e.

htpy equiv. to $M_{1/4}$

$\partial M \times$ interval $\simeq$ h.e., $\partial M \times \{\frac{1}{4}\}$.

(over $\mathbb{Z}$, similar over other R's)

<u>Cor:</u> M (cpct connected m'fold w/ ∂) is ($\mathbb{Z}$-) orientable iff $H_n(M, \partial M) = \mathbb{Z}$.

A fund. class (choice of generator in $H_n(M, \partial M)$) $\Longleftrightarrow$ a choice of orientation.

<u>Thm:</u> (Poincaré duality for manifolds with boundary). M^n cpct with boundary, orientable, fix $[M] \in H_n(M, \partial M)$ (R-coeffs / R-orientations implicit).

$\Rightarrow$ get maps which are isomorphisms

(1) $\quad D_M = (-) \cap [M] : H^{\ell}(M, \partial M) \xrightarrow{\cong} H_{n-\ell}(M)$

(2) $\quad D_M = (-) \cap [M] : H^{\ell}(M) \xrightarrow{\cong} H_{n-\ell}(M, \partial M)$.

The first observation is that (2) follows from (1) and

maps between

Lemma: $\exists$ comm. diagram of duality maps b/w LES's associated to the pair $(M, \partial M)$:

$$
\begin{tikzcd}
\cdots \arrow[r] & H^k(M,\partial M) \arrow[r] \arrow[d, "D_M\,(1)", "\text{by assumption}\ \cong"'] & H^k(M) \arrow[r] \arrow[d, "D_M\,(2)"] & H^k(\partial M) \arrow[r, "\delta^*"] \arrow[d, "D_{\partial M}\ \cong\ \text{b/c } \partial M \text{ is compact.}"] & H^{k+1}(M,\partial M) \arrow[r] \arrow[d, "D_M\,(1)\ \cong\ \text{by assumption}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k}(M,\partial M) \arrow[r, "\partial_*"] & H_{n-k-1}(\partial M) \arrow[r] & H_{n-k-1}(M) \arrow[r] & \cdots
\end{tikzcd}
$$

(part of this lemma: an orientation on M induces one on ∂M, compatible w/

$$\partial_*: H_n(M,\partial M) \to H_{n-1}(\partial M)$$
$[M] \longmapsto$ a choice of fund. class in $H_{n-1}(\partial M)$).

So 5-lemma + (1) $\Rightarrow$ (2).

(Exercise: why is this lemma true?)

Why is (1) true? (1) states that $D_M: H^\ell(M,\partial M) \xrightarrow{\cong} H_{n-\ell}(M)$.

Idea is we want to deduce (1) from P.D. for the noncompact (or not nec. cpct) $\operatorname{int}(M) = \mathring{M}$.

Non-compact P.D. implies:

$$H^\ell_c(\mathring{M}) \underset{\cong}{\xrightarrow{D_M}} H_{n-\ell}(\mathring{M})$$

Now we'll use the fact that $\exists$ an exhaustion of $\mathring{M}$ by compact sets

$$(M_1 \subset M_2 \subset M_3 \subset \cdots) \text{ in } \mathring{M},$$

with each $M_i \subset M_{i+1}$ a homotopy equiv, & each $M_i \underset{\text{homeo.}}{\cong} M$. (uses collar nbhd).

In particular $\{M_i\}$ is cofinal in $\{\text{cpct } K \subset \mathring{M}\}$.

so
$$
\begin{aligned}
H^\ell_c(\mathring{M}) &\cong \varinjlim_i H^\ell(\mathring{M}, \mathring{M}\setminus M_i) \\
&\uparrow \cong \text{ excision} \\
&\varinjlim_i H^\ell(\mathring{M}_{big}, \mathring{M}_{big}\setminus M_i) \\
&\downarrow \cong \quad \text{b/c } \mathring{M} \supseteq M_i \text{ for each } i. \text{ and } \mathring{M}_{big}\setminus M_i \xrightarrow{\simeq} \mathring{M}_{big}\setminus \mathring{M}. \\
&H^\ell(\mathring{M}_{big}, \mathring{M}_{big}\setminus \mathring{M}) \underset{\text{excision}}{\cong} H^\ell(M,\partial M).
\end{aligned}
$$

($M \xrightarrow{\simeq} \mathring{M}_{big}$ ...

$M_{big} = M \cup_{\partial M} \partial M \times [-\varepsilon, 0]$ identified using collar.
(i.e., glue $\partial M \times [-\varepsilon, \varepsilon]$ to $\partial M \times [0,1)$ along $\partial M \times [0,\varepsilon]$).

$(S, \leq)$ direct system, then a subset T of S inherits $\leq$. Say $T \hookrightarrow S$ is cofinal if every s is $s \leq t \in T$.
If $T \subset S$ cofinal, then $\varinjlim_{\alpha \in S} G_\alpha \cong \varinjlim_{\alpha \in T} G_\alpha$.

An exhaustion by cpct sets $\{M_i\}$ is cofinal in $\{\text{cpct sets}\}$ b/c any $K \underset{\text{cpct}}{\subset} M$ is in M_N for some $N \gg 0$ depending on K.
(exercise: why?)

A cpct-exhaustion is

$\longrightarrow M_{big}(M)$.

$K_1 \subset K_2 \subset \cdots \subset \quad S$
w/ K_i cpt, $K_i \subset \mathrm{int}(K_{i+1})$
& $\bigcup K_i = S$.

(i.e., for a manifold-with-boundary, $\boxed{H^\ell_c(\mathrm{int}(M)) \cong H^\ell(M, \partial M)}$).

& moreover, want to check:

$$\begin{tikzcd}
H^\ell_c(\mathring{M}) \arrow[r, "\cong D_M"] \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, phantom, "\circlearrowright"] & H_{n-\ell}(\mathring{M}) \arrow[d, phantom, "\cong" rotate=90] \\
H^\ell(M, \partial M) \arrow[r, "D_M"', "(1)"] & H_{n-\ell}(M)
\end{tikzcd}$$

$\Rightarrow$ (1) is an isomorphism. $\blacksquare$.