Today: fiber bundles, vector bundles, principal bundles

special examples of fiber bundles, with more structure.

the 'fiber' of E at b

Def: A fiber bundle over B is a space E w/ a map $\pi: E \to B$ (continuous), satisfying (local triviality): for every $b \in B$, denoting $E_b := \pi^{-1}(b)$, $\exists$ open $U \ni p$ in B and a map $E|_U := \pi^{-1}(U) \xrightarrow{t} E_b$ such that the map

$$E|_U \xrightarrow{(\pi, t) = \varphi} U \times E_b$$

is a homeomorphism. (note φ fits into a comm. diagram

$$\begin{tikzcd} E|_U \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & & U \times E_b \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

)

proj. to first factor.

Note: any two fibers of a fiber bundle in the same connected component of B must be homeomorphic. We'll often just restrict to a connected B or assume all fibers homeo.

Example: (1) F any space, form

$$\begin{tikzcd} X \times F \arrow[d, "\pi_X"] \\ X \end{tikzcd}$$

trivial fiber bundle w/ fiber F.

(2) covering space

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"] \\ X \end{tikzcd}$$

is a fiber bundle w/ discrete fibers.

(3) (non-discrete, non-trivial example):

$S^3 \subset \mathbb{C}^2$ unit sphere, & consider $\pi: S^3 \longrightarrow \mathbb{C}P^1 = S^2$

$v \longmapsto \{$complex line in $\mathbb{C}^2$ through 0 & $v\}$

'Hopf fibration'

(concretely,

$$\begin{tikzcd} S^3 \arrow[r, hook] \arrow[rr, bend right, "\pi"'] & \mathbb{C}^2 \setminus 0 \arrow[r, "\text{quotient}"] & \mathbb{C}P^1 \end{tikzcd}$$

)

This gives a fiber bundle over S^2 whose fibers are all (S^1)'s. (b/c $\mathrm{span}_{\mathbb{C}}(\overset{\overset{S^3}{\cup}}{v}) = \mathrm{span}_{\mathbb{C}}(\overset{\overset{S^3}{\cup}}{\overbrace{e^{i\theta}v}})$).

This is not a trivial fiber bundle (i.e. not isomorphic to one): $S^3 \neq S^2 \times S^1$ (e.g., H_1's are different)

(4) $V_k(\mathbb{R}^n)$ Stiefel manifold

exercise from 535a: show this is a smooth manifold.

$= \{$orthogonal k-frames in $\mathbb{R}^n\} = \{A \in \mathrm{Mat}(n \times k) \mid AA^T = Id_k\}$.

This is a compact manifold. (How to see this? To start, observe $O(n)$ acts on $V_k(\mathbb{R}^n)$ by composition: transitive action, & isotropy group of basepoint $\{e_1,\dots,e_k\}$ is $I_k \times O(n-k)$. $\xRightarrow[\text{can show}]{\text{using this,}}$

$$V_k(\mathbb{R}^n) = \underset{\nearrow\ \text{compact}}{O(n)} \Big/ I_k \times O(n-k)$$

($\Rightarrow$ Hausdorff, cpct).

- Get a fiber bundle (fibration) $O(n) \longrightarrow V_k(\mathbb{R}^n)$ with fiber $O(n-k)$. (why? locally trivial?) e.g., why

  e.g., $V_1(\mathbb{R}^n) = S^{n-1}$, so in particular $O(n) \to S^{n-1}$ w/ fiber $O(n-1)$.

- Forget last $(k-1)$ vectors: $V_k(\mathbb{R}^n) \longrightarrow V_1(\mathbb{R}^n) = S^{n-1}$. with fiber at $v \in S^{n-1}$ the collection of $(k-1)$ tuples of orthogonal frames that are orthogonal to v, i.e., $(k-1)$-orthogonal frames of $T_v S^{n-1}$.

The basic results that allow for us to show examples in (4) are fiber bundles (& many other examples) are:

Thm: (Ehresmann): Say E, B smooth manifolds, $\pi: E \to B$. smooth map. If π is

- proper (i.e., $\pi^{-1}(\text{cpct.})$ is cpct)
- submersion (means $d\pi_x : T_xE \to T_xB$ surjective for all x).

then $\pi: E \to B$ is a fiber bundle.

Using this, can prove:

Prop: G Lie group, and $K \subseteq H \subseteq G$ closed subgroups (so K, H also Lie groups) then the projection map

$$\begin{tikzcd} G/K \arrow[r] & G/H \\ g+K \arrow[r, mapsto] & g+H. \end{tikzcd}$$

is a fiber bundle with fibers isomorphic to H/K.

can apply this general result to get examples in (4), and many others.

E.g.,

(5) Grassmannians.

$G_k(\mathbb{R}^n)$ (or $Gr_{\mathbb{R}}(k,n)$) $:= \{ V \subset \mathbb{R}^n \mid V \text{ a real linear } k\text{-dim'l subspace} \}$.

subspace

$$G_1(\mathbb{R}^{n+1}) := \mathbb{R}P^n.$$

(There's also a complex version:

$$G_k(\mathbb{C}^n) := \{ V \subset \mathbb{C}^n \mid V \text{ a cplx-linear } k\text{-dim'l subspace} \}.$$

w/ $G_1(\mathbb{C}^{n+1}) := \mathbb{C}P^n$., w/ same construction.)

can explicitly construct as

$$G_k(\mathbb{R}^n) = \{ \text{linearly independent } k\text{-tuples in } \mathbb{R}^n \} \Big/ GL(k,\mathbb{R})$$

open subset of $(\mathbb{R}^n)^k$.

applying $GL(k,\mathbb{R})$ to a tuple gives same span.

equipped w/ quotient topology,

can also construct as

$$= \{ \text{orthonormal } k\text{-tuples in } \mathbb{R}^n \} \Big/ O(k)$$

$k\times n$ matrices A w/ $AA^T = Id_k$.

$$= \{ \text{orthonormal } n\text{-tuples in } \mathbb{R}^n \} \Big/ O(k)\times O(n-k)$$

$$= O(n) \Big/ O(k)\times O(n-k)$$

can check again that $G_k(\mathbb{R}^n)$ is a cpct, Hausdorff manifold.

The Prop above implies: $V_k(\mathbb{R}^n) \longrightarrow G_k(\mathbb{R}^n)$ is a fiber bundle w/ fibers $O(k)$.

$$\{v_1, \dots, v_k\} \longmapsto \operatorname{span}(v_1, \dots, v_k)$$

As we'll see, many of the above examples have the structure of principal bundles.

Vector bundles

a type of fiber bundle where all fibers are vector spaces (& trivializations are compat. w/ this structure).

X a space.

Def: A real vector bundle over X is

(i) a space E

(ii) a continuous $\pi: E \to X$

(iii) a real vector space structure on each $E_x := \pi^{-1}(x)$, $x \in X$.

Satisfying (local triviality):

for every $x_0 \in X$, $\exists$ a nhood $U \ni x_0$ in X and a homeo. for some n

$$\begin{tikzcd} E|_{U} = \pi^{-1}(U) \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & \circlearrowleft & U \times \mathbb{R}^n \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

(proj. to first factor)

s.t. $\varphi|_{E_x} : E_x \xrightarrow[(\text{by } \circlearrowleft)]{} \{x\}\times\mathbb{R}^n \cong \mathbb{R}^n$ is a real linear isomorphism, for each $x \in U$.

Similarly, have notion of a complex vector bundle: (replace real by complex & $\mathbb{R}^n$ by $\mathbb{C}^n$).

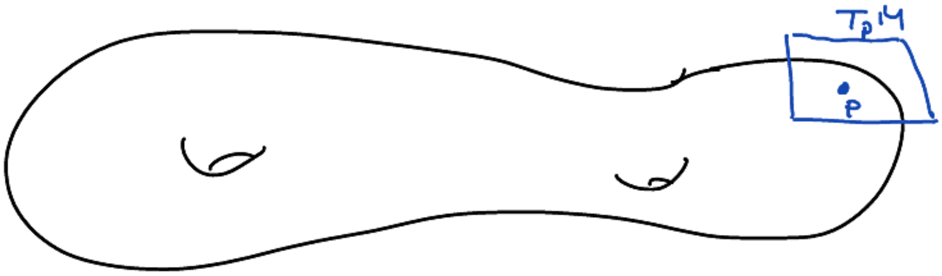
Examples:

(i) $X \times \mathbb{R}^n =: \underline{\mathbb{R}}^n$ equipped w/ $\underset{\pi_X}{\overset{\pi}{\shortparallel}} : X \times \mathbb{R}^n \longrightarrow X$ (projection to X)

trivial vector bundle.

(ii) M any smooth (C^∞) manifold, then its tangent bundle $TM \xrightarrow{\pi} M$ (fiber at $p \in M$ is T_pM tangent space) is a vector bundle.

e.g., if $M \subset \mathbb{R}^N$



(so are T^*M, $\Lambda^k T^*M$, etc.)

(iii) Tautological vector bundles on Grassmannians

Define $E_{taut} \xrightarrow{\pi} Gr_k(\mathbb{R}^n)$ by:

(similarly $E_{taut} \longrightarrow Gr_k(\mathbb{C}^n)$ tautological complex vec. bundle)

$$E_{taut} \subseteq Gr_k(\mathbb{R}^n) \times \mathbb{R}^n$$
$$\shortparallel$$
$$\{ (\mathbf{X}, v) \mid \mathbf{X} \in Gr_k(\mathbb{R}^n),\ v \in \mathbf{X} \}$$

and $\pi(\mathbf{X}, v) := \mathbf{X}$.

Observe: $(E_{taut})_{\mathbf{X}} := \pi^{-1}(\mathbf{X}) = \overset{\text{the point}}{\{\mathbf{X}\}} \times \overset{\text{the subspace of } \mathbb{R}^n}{\mathbf{X}} \cong \mathbf{X}$, i.e., has a linear structure.

Local triviality?

Choose a surjection $\alpha : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^k$ (linear).

true whenever $\mathbf{X} \cap \overbrace{\ker(\alpha)}^{(n-k)\text{-dim'l}} = \{0\}$

Define $U_\alpha := \{ X \in Gr_k(\mathbb{R}^n) \mid \alpha|_X : X \longrightarrow \mathbb{R}^k \text{ is an isomorphism} \}$

(open dense subset, and $\{U_\alpha\}_{\alpha \in Surj(\mathbb{R}^n, \mathbb{R}^k)}$ cover $Gr_k(\mathbb{R}^n)$)

On U_α have a trivialization

$$\begin{tikzcd} E|_{U_\alpha} \arrow[r, "\varphi_\alpha"] & U_\alpha \times \mathbb{R}^k \\ (X, v) \arrow[r, mapsto] & (x, \alpha|_X(v)). \end{tikzcd}$$

$(x \in U_\alpha, v \in X)$

check (exercise):
- homeomorphism, compat w/ projections.
- linear in each fiber.

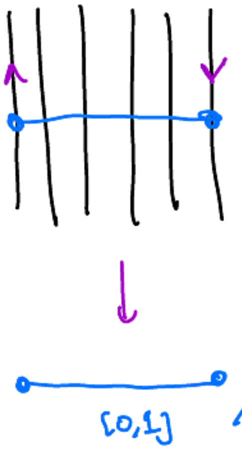
<u>Def</u>: The rank of $E \xrightarrow{\pi} X$ is $\dim_{\mathbb{R} \text{ or } \mathbb{C}} (E_x)$, provided this number is constant in X.
(over $\mathbb{R}$ or $\mathbb{C}$) $\uparrow$ any x

(know it has to be locally constant b/c local triviality, we'll usually assume global constancy so we can talk about rank)

(real or complex) <u>line bundle</u>: vector bundle of (real or complex) rank $= 1$.

e.g., tautological bundles over $Gr_k(\mathbb{R}^n)$ $Gr_k(\mathbb{C}^n$ when $k=1$ give:
- $L_{taut} \longrightarrow \mathbb{CP}^n$ tautological (complex) line bundle
- $L_{taut} \longrightarrow \mathbb{RP}^n$ tautological (real) line bundle

<u>subexample/exercise</u>: Look at $L_{taut} \longrightarrow \mathbb{RP}^1 \cong S^1$ & verify L_{taut} is $\cong$ <u>Möbius bundle</u>:



$[0,1] \times \mathbb{R} \big/ (0,v) \sim (1,-v)$

$\big/ 0 \sim 1.$

& verify L_{taut} is <u>not trivial</u>.

Def: An *isomorphism of vector bundles* $E \xrightarrow{\pi_E} X$, $F \xrightarrow{\pi_F} X$ is a homeomorphism,

compat. w/ projections:

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & \circlearrowleft & F \arrow[dl, "\pi_F"] \\ & X & \end{tikzcd}$$

, such that $\varphi|_{E_x} : E_x \to F_x$ is a linear isomorphism for each $x \in X$.

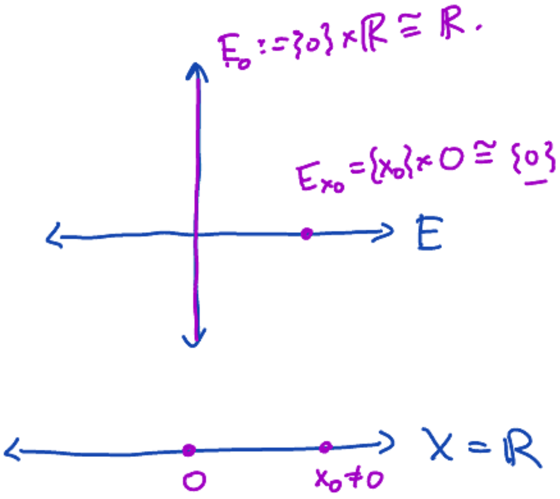
Automorphisms are self-isomorphisms.

E.g., $\mathrm{Aut}(\underset{\text{vec. bdle over } X}{\underline{\mathbb{R}}^k}) = \mathrm{Maps}(X, GL(k,\mathbb{R}))$.

Non-example of a vector bundle (also not a fiber bundle):

$E = \{xy = 0\} \subseteq \mathbb{R}^2$ = x-axis ∪ y-axis

$$\begin{tikzcd} (x,y) \arrow[d, mapsto] & E \arrow[d, "\pi_x"] \\ x & \mathbb{R} \end{tikzcd}$$



(this example can be viewed as, in a suitable sense, a *sheaf*):

Principal bundles

G a topological group, X a space.

Def: A *principal G-bundle* (or a principal bundle w/ *structure group* G) over X is a fiber bundle $\pi : P \to X$, along with a right action of G $P \times G \to P$, such that $\pi : P \to X$ is the quotient by this action, and (implies G preserves each P_x & hence $P|_U$.)

(*local triviality*) $\exists$ an open cover $\mathcal{U}$ of X s.t. for every $U \in \mathcal{U}$, $\exists$ a trivialization ("local trivialization along U")

$$\begin{tikzcd} P|_U^{\circlearrowleft G} \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\underset{\text{map from above}}{\pi}"'] & \circlearrowleft & U \times G^{\circlearrowleft G} \arrow[dl, "\pi_U \leftarrow \text{proj. to } U"] \\ & U & \end{tikzcd}$$

which is G-equivariant; i.e., if $\varphi(p) = (z, g_0)$ then $\varphi(pg) = (z, g_0 g)$.

(note: G acts freely on P, & each $P_x \cong G$, as spaces w/ G action (but no canon. group str. on P_x)

Obs: If $\pi: E \to X$ is a vector bundle of rank k, $\exists$ an associated principal $GL(k,\mathbb{R})$ bundle $\tilde{\pi}: P \to X$, defined as $P = \{(x, v_1, \ldots, v_k) \mid x \in X,\ (v_1, \ldots, v_k) \text{ basis for } E_x\}$.

"*frame bundle*", Frame(E).

$GL(k,\mathbb{R})$ acts on P by "change of basis" action, local triviality follows from local triviality of $E \to X$.

It turns out one can naturally go back from Frame(E) to E, as a special case of a more general construction that associates

$$(P := \text{principal } G \text{ bundle},\ \underset{\text{representation}}{G \to GL(V)}) \longmapsto P \times_G V \text{ associated vector bundle.}$$

Applying this to

$$(\text{Frame}(E),\ GL(k) \xrightarrow{id} GL(k)) \quad \text{produces } E.$$

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Operations on principal bundles:

$P \xrightarrow{\pi} X$ principal G-bundle, F any top. space w/ a left G action $G \times F \to F$

($G \to \text{Homeo}(F)$ map of top. groups)

$\rightsquigarrow$ can form the *associated fiber bundle*

$$P \times_G F := P \times F/\sim \quad \text{where } (zg, f) \sim (z, gf). \ \forall g, z, f.$$

$\pi: P \times_G F \to X$ defined by $\pi([z,f]) := \pi(z)$ (check well-defined);

fibers non-canonically isomorphic to F, & locally trivial (check: uses local triviality of P).

If the action has 'more structure', the associated fiber bundle will have more structure too.

e.g.,
- If $F = V$ a vector space (over $\mathbb{R}$ or $\mathbb{C}$) and $G \times V \to V$ is a *linear action* (meaning $G \to GL(V) \subset \text{Homeo}(V)$),

then $P \times_G V$ is a vector bundle of rank $= \dim(V)$, w/ fibers all (non-canonically) isomorphic to V.

- If have a map of top. groups $G \to H$ (e.g., continuous group hom.), then $P \times_G H$ is a *principal H-bundle*.

induces an action $G \times H \to H$.

Let's give some examples of this construction.

Note: Have the tautological action $GL(\mathbb{R}^k)\times\mathbb{R}^k \longrightarrow \mathbb{R}^k$ $\quad$ $\left(GL(\mathbb{R}^k) \xrightarrow{id} GL(\mathbb{R}^k) \right).$

$(T, v) \longmapsto T(v)$

using this) action

Claim: If $\begin{tikzcd} E \arrow[d, "\pi"'] \\ X \end{tikzcd}$ any vector bundle $\rightsquigarrow$ $\begin{tikzcd} Frame(E) \arrow[d] \\ X \end{tikzcd}$ principal $GL(\mathbb{R}^k)$ bundle $\rightsquigarrow$ $Frame(E)\times_{GL(\mathbb{R}^k)} \mathbb{R}^k$

Then $Frame(E)\times_{GL(\mathbb{R}^k)}\mathbb{R}^k \cong E.$

In fact, (exercise): The following are inverse operations

$$\begin{tikzcd} (*) \quad \left\{ \begin{array}{c} \text{Vector bundles of} \\ \text{rank } k \\ \text{on } X \end{array} \right\} \arrow[r, shift left, "Frame"] & \left\{ \begin{array}{c} \text{Principal } GL(k,\mathbb{R}) \\ \text{bundles on } X \end{array} \right\} \arrow[l, shift left, "(-)\times_{GL(\mathbb{R}^k)} \mathbb{R}^k"] \end{tikzcd}$$

tautological action.

In particular by applying (*), given a representation $GL(\mathbb{R}^k) \to GL(\mathbb{R}^m)$, we get an associated operation

$$\begin{tikzcd} \{\text{rank } k \text{ vector bundles}\} \arrow[r, dashed] \arrow[d] & \{\text{rank } m \text{ vector bundles}\} \\ \text{Principal } GL(k)\text{-bundle} \arrow[r, "\text{assoc. bundle}"'] & \text{Principal } GL(m)\text{-bundle} \arrow[u] \end{tikzcd}$$

Ex: (1) $\overset{GL(\mathbb{R}^k)}{\overset{\shortparallel}{GL(k,\mathbb{R})}}$ acts on $\mathbb{R}$ by $GL(k,\mathbb{R}) \longrightarrow GL(1,\mathbb{R}) = \mathbb{R}\backslash 0$

$A \longmapsto \det(A)$

$\rightsquigarrow$ get for any rank k $E\to X$ an associated line bundle $\det(E) \longrightarrow X$. (note: this coincides w/ "$\begin{tikzcd} \Lambda^{top} E \arrow[d] \\ X \end{tikzcd}$").

(2) Consider $GL(k,\mathbb{R})$ acting on $\mathbb{R}^k$ via

$$(A, \vec{v}) \longmapsto (A^{-1})^T\vec{v}.$$

The associated vector bundle (starting from E) is called the dual vector bundle E^*.
(similar constructions work over $\mathbb{C}$)

Other operations on vector bundles: (over $\mathbb{R}$ or $\mathbb{C}$)

- Pullback: Given a vector bundle $\begin{tikzcd} E \arrow[d, "\pi"] \\ Y \end{tikzcd}$ and a continuous map $f: X\to Y$, get a vector bundle $\begin{tikzcd} f^*E \arrow[d, "f^*\pi"] \\ X \end{tikzcd}$, along with a map (lying over f)

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*\pi"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(linear on each fiber)

by definition, $f^*E := \{(x,e) \mid f(x) = \pi(e)\} \subseteq X \times E$, and $(f^*\pi)(x,e) = x$.

("$X \times_Y E$" or "$X \times_{(f,\pi)} E$")

Note: $(f^*E)_x := E_{f(x)}$. (a vector space).

Locally trivial? (exercise).

Note: • We can also pull back principal bundles via the same construction (replace E w/ P), & the G action is inherited from G action on $X \times P \supseteq X \times_Y P = f^*P$.

• special case: $X \overset{i}{\subset} Y$ inclusion of subset, then $i^*E = E|_X$ $(= \pi^{-1}(X))$.

• Cartesian product of vector bundles (or principal bundles)

rank m rank n

if
$$\begin{tikzcd} E \arrow[d, "\pi_E"] & F \arrow[d, "\pi_F"] \\ X & Y \end{tikzcd}$$
(resp.
$$\begin{tikzcd} P \circlearrowleft G \arrow[d, "\pi_P"] & Q \circlearrowleft H \arrow[d, "\pi_Q"] \\ X & Y \end{tikzcd}$$
), then

$$\begin{tikzcd} E \times F \arrow[d, "(\pi_E, \pi_F)"] \\ X \times Y \end{tikzcd}$$
(resp.
$$\begin{tikzcd} P \times Q \arrow[d, "(\pi_P, \pi_Q)"] \\ X \times Y \end{tikzcd}$$
) is a vector bundle of rank m+n. (resp. principal $G \times H$ bundle).

(exercise)

• "fiberwise direct sum" of vector bundles (whitney sum):

Given
$$\begin{tikzcd} E \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"'] \\ X & X \end{tikzcd}$$
, first take
$$\begin{tikzcd} E \times F \arrow[d, "(\pi_E, \pi_F)"] \\ X \times X \end{tikzcd}$$
, then

define $E \oplus F := \Delta^*(E \times F)$, where $\Delta : X \to X \times X$ diagonal embedding.
$x \longmapsto (x,x)$

check: $(E \oplus F)_x := E_x \oplus F_x$.

• We can similarly define "fiberwise" operations $E \otimes F$, $\mathrm{Hom}_{\mathbb{R}}(E,F)$; easiest way to see this is as follows:

or $\mathrm{Hom}_{\mathbb{C}}$ if $\mathbb{C}$-vec. bdles

rank m rank n

starting with
$$\begin{tikzcd} E \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ X & X \end{tikzcd}$$
, let
$$\begin{tikzcd} P \arrow[d, "\pi_P"'] & Q \arrow[d, "\pi_Q"] \\ X & X \end{tikzcd}$$
be associated frame bundles over X.

P has structure group $G := GL(m, \mathbb{R})$

Q " " $H := GL(n, \mathbb{R})$.

Form $\Delta^*(P \times Q) := P \times_X Q$ · a principal $G \times H$ bundle over X.

Observe that $G\times H = GL(m,\mathbb{R})\times GL(n,\mathbb{R})$ acts naturally on

- $\mathbb{R}^m \oplus \mathbb{R}^n$ by $(g,h)(v\oplus w) = gv \oplus hw$
- $\mathbb{R}^m \otimes \mathbb{R}^n$ by $(g,h)(v\otimes w)$ is $gv\otimes hw$
- $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m,\mathbb{R}^n)$ $(g,h)(T) = h\circ T\circ (g^{-1})^T$

We call the associated bundles $E\oplus F$, $E\otimes F$, $\mathrm{Hom}_{\mathbb{R}}(E,F)$ ← resp. $\mathrm{Hom}_{\mathbb{C}}(-,-)$.

check: agrees w/ def'n above. The fiber at each $x\in X$ is $E_x\oplus F_x$, $E_x\otimes F_x$, $\mathrm{Hom}_{\mathbb{R}}(E_x,F_x)$ respectively.

- the dual bundle can be realized as $E^* = \mathrm{Hom}_{\mathbb{R}}(E,\underline{\mathbb{R}})$.

Def: A section of a fiber bundle $\begin{matrix} Q \\ \downarrow\pi \\ X\end{matrix}$ is a map $s: X\to Q$ with $\pi\circ s = \mathrm{id}_X$. ☆

denoted
$$\begin{tikzcd} Q \arrow[d, "\pi"] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$$

☆ ⇒ $s(x) = (x, s_x)$ where $s_x\in Q_x$

(thinking of Q set theoretically as $\coprod_{x\in X} Q_x$).

Thm: A principal bundle is trivial iff it has a section.

(rmk: in contrast, while it is true a line bundle is trivial $\Leftrightarrow$ $\exists$ non-zero section, not nec. true for higher rank vec. bundles)

(E vec. bundle rank k ⇝ $\mathrm{Frame}(E)$ is trivial iff $\exists$ a section $X\to \mathrm{Frame}(E)$ ⇝ E is trivial iff $\exists$ a k-tuple of sections which form a frame at each point x (i.e., a basis for each fiber)).

Pf: $\Rightarrow$ ✓ b/c
$$\begin{tikzcd} X\times G \arrow[d] \\ X \arrow[u, bend right, "s(x)=(x,\mathrm{id})"'] \end{tikzcd}$$

$\Leftarrow$ Say $\exists$ s:
$$\begin{tikzcd} P \arrow[d, "\pi"] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$$
. Then define
$$\begin{tikzcd} X\times G \arrow[rr, "\varphi"] \arrow[dr, "\pi_X"'] & & P \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$
, by $\varphi(x,g) = s(x)\cdot g$.

a map of principal bundles (i.e., φ G-equiv.)

φ is automatically an iso. by next lemma. ∎

Lem: Any non-trivial morphism of G-bundles
$$\begin{tikzcd} P_0 \arrow[rr, "f"] \arrow[dr] & G & P_1 \arrow[dl] \\ & X & \end{tikzcd}$$
(i.e., G-equiv.) is an isomorphism.

Pf: Special case
$$\begin{tikzcd} P_0 = X\times G \arrow[d, "\pi_X"] \\ X \end{tikzcd}, \quad \begin{tikzcd} P_1 = X\times G \arrow[d, "\pi_X"] \\ X \end{tikzcd}$$
&
$$\begin{tikzcd} f: P_0 \arrow[r] \arrow[dr] & P_1 \arrow[d] \\ & X \end{tikzcd}$$
$\overset{\text{equivariance}}{\Rightarrow} f(x,g) = (x, g\,h(x))$ for some $h: X\to G$.

But now this map has inverse $(x,g)\longmapsto (x, g(h(x))^{-1})$.

Since a general P_0, P_1 are locally trivial, this argument applies f is an iso. in a nbhd of any x, hence everywhere. ∎ .

(an inner product on V is an element of $(V\otimes V)^* \ni g$ s.t. the map $\langle -,-\rangle : V\times V \to V\otimes V \to \mathbb{R}$ satisfies --)

Inner products on vector bundles:

An inner product on a vector bundle $\overset{E}{\underset{X}{\downarrow}}$ is a section (g) of $(E\otimes E)^*$,

s.t. the associated pairing $\langle -,-\rangle_x$ on E_x defined by $\langle v,w\rangle_x := g_x(v\otimes w)$

is an inner product (pos definite, symmetric bilinear).

Can think of as a collection of $\langle -,-\rangle_x$ on each E_x "varying continuously" (in sense g is a continuous section)

$\Rightarrow$ If s, t are (continuous) sections, then

$x \longmapsto \langle s_x, t_x\rangle_x$ is continuous.

OR $\langle -,-\rangle \in$ "$\Gamma(\mathrm{Bilinear}(E\times E, \mathbb{R}))$".

c.f. 535a or Hatcher

Lem: An inner product exists (at least if X is paracompact, i.e., admits partitions of unity)

Sketch: Given a cover $\{U_\alpha\}$ over which E is loc. trivial, $\exists$ an inner product $\langle -,-\rangle_\alpha$ on each $E|_{U_\alpha}$ b/c $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ (& use $\langle -,-\rangle_{\text{Euclidean}}$ on $\mathbb{R}^k$).

Then if $\{\varphi_\alpha\}$ is a partition of 1 subordinate to $\{U_\alpha\}$, we claim

$\sum \varphi_\alpha \langle -,-\rangle_\alpha$ gives an inner product on E. (exercise). ∎ .

--- 3/8/2021

Q: If a vector bundle comes equipped with an inner product, how can I understand this in terms of principal bundles?

Def: $P \to B$ principal G-bundle, $H \subseteq G$ subgroup. Say P has a reduction of structure group to H iff P is isomorphic to $\tilde{P} \times_H G$ for some $\tilde{P} \to B$ principal H-bundle. A choice of reduction is a choice of such $\tilde{P}$.

Lemma: Given a vector bundle $E \to B$, an inner product on $E \longleftrightarrow$ a choice of reduction of structure group of $\mathrm{Frame}(E)$ to $O(n)$ (from $GL(n,\mathbb{R})$).

Idea: Given $\langle -,-\rangle$ on E, can consider $\mathrm{OFrame}(E) = \{(x, v_1, \dots, v_k) \mid x \in B,\ v_1, \dots, v_k$ an orthonormal frame of E_x w.r.t. $\langle -,-\rangle_x\}$.

Claim: $\mathrm{OFrame}(E) \times_{O(n)} GL(n,\mathbb{R}) \cong \mathrm{Frame}(E)$. (exercise).

This defines a map

$$\begin{tikzcd}\left\{\begin{array}{c}\text{Vector bundles} \\ \text{w/ inner product}\end{array}\right\} \arrow[r, shift left, "\text{OFrame}"'] & \left\{\text{principal } O(n)\text{-bundles}\right\} \arrow[l, shift left, "?"']\end{tikzcd}$$

$P \mapsto P \times_{O(n)} (\mathbb{R}^n, \langle -,- \rangle_{\text{Eucl.}})$ ⟵ (claim: if $G \curvearrowright \mathbb{R}^n$ by isometries, then the Euclidean inner product on $\mathbb{R}^n$ induces an inner product on each fiber of $P \times_G \mathbb{R}^n$.)

In general many extra structures on vector bundles can be described via a reduction of structure group of the associated principal bundle.

(e.g., w/o proof:
- An orientation on E^k $\longleftrightarrow$ Reduction of structure grp of Frame(E) to $GL^+(k,\mathbb{R}) \subseteq GL(k,\mathbb{R})$
- A complex structure on $\underset{\text{real vec. bundle}}{E^{2k}}$ $\longleftrightarrow$ " " " to $GL(k,\mathbb{C}) \subseteq GL(2k,\mathbb{R})$)

Homotopy invariance of pullbacks

Lemma: If $f_0, f_1 : X \longrightarrow Y$ are homotopic maps, $E \xrightarrow{\pi} Y$ vector bundle or principal G-bundle, and say X is paracompact. Then $f_0^* E \cong f_1^* E$ as (vector/principal) bundles over X.

Rmk: Same is true for arbitrary fiber bundles.

To prove Lemma, we'll make use of an important property satisfied by such bundles over paracompact spaces, homotopy lifting property (HLP). (already appears in e.g., covering space theory) ⟵ $A \subseteq X$ subspace. "with respect to (X,A)."

Def: A map $\begin{array}{c} E \\ \downarrow \pi \\ B \end{array}$ satisfies HLP with respect to X (rel A) if,

given a homotopy $F : X \times I \longrightarrow B$ and a lift

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ X \times 0 \arrow[ur, "\tilde{f}_0"] \arrow[r, "f_0"'] & B \end{tikzcd}$$

$f_0 = F|_{X \times 0}$

of $f_0 = F(-,0)$, (i.e., $\pi \circ \tilde{f}_0 = f_0$),

(and a lift $\tilde{G} : A \times I \longrightarrow E$ of $F|_{A \times I}$, so $\pi \circ \tilde{G} = F|_{A \times I}$, agreeing w/ $\tilde{f}_0|_A$ along $A \times 0$).

Then, $\exists$ a homotopy $\tilde{F} : X \times I \to E$ lifting F (i.e., $\pi \circ \tilde{F} = F$) and agreeing with $\tilde{f}_0 = F(-,0)$ on $X \times 0$.

(and further agreeing with $\tilde{G}$ when restricted to $A \times I$).

What we need is:

Thm: (Hatcher prop 4.48 + references that follow):

A fiber bundle $E \xrightarrow{\pi} B$ has HLP for all (X,A) if B is paracompact.

(Hatcher proves explicitly that even if B not paracompact $E \xrightarrow{\pi} B$ has HLP for all CW pairs).

Rmk: A weaker condition than requiring $E \xrightarrow{\pi} B$ to a fiber bundle is requiring it to satisfy HLP for all CW pairs (X,A), equivalently (by iteration) for all $(D^n, \partial D^n)$ $\forall n$. This is called having a Serre fibration, & suffices for many purposes.

Proof of homotopy invariance lemma: (Recall have $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$, $f_0, f_1 : X \to B$)

Let $F : X \times I \to Y$ be the homotopy (so $f_0 = F(-,0)$, $f_1 = F(-,1)$) and consider the pullback

$$\begin{tikzcd} F^*E \arrow[d] \\ X \times I \end{tikzcd}$$

We want to show that $F^*E|_{X\times\{0\}} \cong F^*E|_{X\times\{1\}} = f_1^*E$.

$$\underset{f_0^*E}{\overset{\shortparallel}{F^*E|_{X\times\{0\}}}}$$

Let $p : X \times I \to X$ projection to X.

It is sufficient to show $p^*f_0^*E \cong F^*E$ as (vector, principal) bundles over $X \times I$.

$$☆ \quad \begin{tikzcd} p^*f_0^*E \arrow[rr, phantom, "\cong"] \arrow[dr] & & F^*E \arrow[dl] \\ & X \times I & \end{tikzcd}$$

(why? restricting to $X \times \{1\}$, we'd get: $f_0^*E \cong f_1^*E$ as desired).

(Specifying the above ☆ amounts to exhibiting an iso for each $x \in X$, $t \in [0,1]$,

$$\begin{tikzcd} (p^*f_0^*E)_{(x,t)} \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\wr\wr"] & (F^*E)_{(x,t)} \arrow[r, phantom, "="] & E_{F(x,t)} \arrow[r, phantom, "="] & E_{f_t(x)} \\ (f_0^*E)_x \arrow[d, phantom, "\shortparallel"] & & & \\ E_{f_0(x)} \arrow[uurrr, dashed] & & & \end{tikzcd} \quad (f_t := F(-,t)).$$

$\uparrow$ coincides w/ $E_{f_0(x)}$ when $t=0$.

(continuously varying in x, t).

)

Consider the fiber bundle

- $$\begin{tikzcd} P = \mathrm{Hom}_G(p^*f_0^*E, F^*E) \arrow[d] \\ X \times I \end{tikzcd}$$ of fiberwise maps (in case E is a principal bundle;)

  $\uparrow$ check: principal G-bundle.

  note a section gives a map of principal bundles $p^*f_0^*E \to F^*E$, which is automatically an iso!

OR

- $$\begin{tikzcd} P = \mathrm{Iso}_{\mathbb{R}}(p^*f_0^*E, F^*E) \arrow[d] \\ X \times I \end{tikzcd}$$ (sub-bundle of $\mathrm{Hom}_{\mathbb{R}}(-,-)$ consisting of fiberwise linear isomorphisms)

  $\leftarrow$ check: this is a principle $GL(k,\mathbb{R})$-bundle, $k = \mathrm{rank}(G)$.

check: this is indeed a fiber bundle, and a section gives precisely the bundle isomorphism

$$p^*f_0^*E \cong F^*E$$ we want.

Observe $P|_{X\times\{0\}}$ has a preferred section:

$$\begin{tikzcd} P|_{X\times\{0\}} \arrow[d] \\ X\times 0 \arrow[u, bend right, "s:(x,0)\mapsto(x,0,\mathrm{id})"'] \end{tikzcd}$$

In other words, the homotopy

$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ X\times I \arrow[r, "\mathrm{id}"] & X\times I \end{tikzcd}$$

has a lift $\widetilde{\mathrm{id}}_0$ along $X\times 0$.

By HLP for $P \to X\times I$ (since $X / X\times I$ are paracompact), we can therefore find a lift of id extending the lift $\widetilde{\mathrm{id}}_0$ along $X\times 0$.

$\Rightarrow p^*f_0^*E \cong F^*E \underset{\substack{\text{restrict}\\ \text{to } X\times 1}}{\Rightarrow} f_0^*E \cong f_1^*E$. $\boxtimes$.

Some consequences of the homotopy invariance property:

Lemma $\Leftrightarrow$ For any $X \to Y$, the map $f^*: \{\text{principal/vector bundles on } Y\}/_{\text{iso.}} \longrightarrow \{\text{principal/vec. bdles on } X\}/_{\text{iso.}}$

only depends on $[f] \in [X, Y]$.

If we denote by $\mathrm{Bun}_G(X) := \{\text{principal } G\text{-bundles on } X\}/_{\text{iso.}}$

$\mathrm{Vect}_k(X) := \{\text{rank } k \text{ vec. bundles on } X\}/_{\text{iso.}}$,

$\Rightarrow$ $\mathrm{Bun}_G(-)$ and $\mathrm{Vect}_k(-)$ are (contravariant) "homotopy functors". (akin to $H^k(-)$).

In particular:

Cor: Over a contractible space, any vec. bundle resp. principal bundle is trivial.

Pf: X contractible, and $x_0 \overset{i}{\hookrightarrow} X$ any point. Then $j: X \to x_0$ (projection) is homotopy inverse, i.e., $ij \overset{\exists \text{ homotopy}}{\simeq} \mathrm{id}_X$ & $ji \simeq \mathrm{id}_{x_0}$ (of course $ji = \mathrm{id}_{x_0}$).

$$\begin{tikzcd} \Rightarrow j^*: \mathrm{Bun}_G(x_0) \arrow[r, "\cong"] & \mathrm{Bun}_G(X) \\ \{x_0\times G\} \arrow[u, phantom, "\shortparallel"] \arrow[r, mapsto, "\text{calculate}"] & \{X\times G\} \end{tikzcd}$$ $\boxtimes$.

$$\begin{tikzcd} (\text{OR } \mathrm{Vect}_k(x_0) \arrow[r, "\cong"] & \mathrm{Vect}_k(X)\ ) \\ \{x_0\times \mathbb{R}^k\} \arrow[u, phantom, "\shortparallel"] \arrow[r, mapsto, "\text{calculate}"'] & \{X\times \mathbb{R}^k\} \end{tikzcd}$$

(We used the more genl Cor: that if $f: X \to Y$ is a homotopy equivalence, then

$$(f)^*: \quad \begin{array}{l} Bun_G(Y) \xrightarrow{\cong} Bun_G(X) \\ Vect_k(Y) \xrightarrow{\cong} Vect_k(X) \end{array} \qquad).$$

Clutching functions:

$E \xrightarrow{\pi} B$ fiber bundle.

Fix a trivializing cover $\{U_\alpha\}_{\alpha \in I}$ of B, along with trivializations

$$\begin{tikzcd} \psi_\alpha : U_\alpha \times F \arrow[rr, "\cong"] \arrow[dr] & & E|_{U_\alpha} \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

On $U_\alpha \cap U_\beta$, comparing trivializations gives us a map over $U_\alpha \cap U_\beta$ of fiber bundles,

$$\begin{tikzcd} (U_\alpha \cap U_\beta) \times F \arrow[r, "\psi_\beta"] \arrow[dr] \arrow[rr, bend left, "(p, f) \longmapsto (p, \phi_{\alpha\beta}(p)(f))"] & E|_{U_\alpha \times U_\beta} \arrow[r, "\psi_\alpha^{-1}"] \arrow[d] & (U_\alpha \cap U_\beta) \times F \arrow[dl] \\ & U_\alpha \cap U_\beta & \end{tikzcd}$$

determined by a map $\phi_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow Homeo(F)$, called the clutching functions of E w.r.t. $\{U_\alpha\}$.

- If E is a vector bundle, by using local trivializations of E as a vector bundles, the clutching functions land in $GL(\mathbb{R}^k) \subseteq Homeo(\mathbb{R}^k)$.
- principal G bundle, clutching functions can be made to take values in G by using a cover trivializing the bundle as principal bundle.

The group the clutching fcns. take value in, $G \subset Homeo(F)$, is called the structure group of the bundle.

The cover $\{U_\alpha\}$ & clutching functions in fact determine the bundle completely:

Given B, a cover $\{U_\alpha\}_{\alpha \in I}$ of B, a space F, a group G which acts on F (i.e., $G \to Homeo(F)$), can form a fiber bundle

$$E = \bigcup_\alpha U_\alpha \times F \Big/ \sim$$ where we identify, for each α, β, $x \in U_\alpha \cap U_\beta$, $(x,f) \in U_\alpha \times F$ with $(x, \phi_{\beta\alpha}(x)(f)) \in U_\beta \times F$.

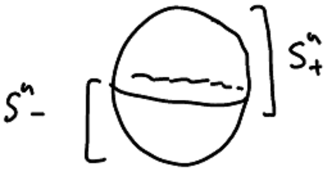
with $\pi : E \xrightarrow[\text{factor}]{\text{projection to left}} \bigcup U_\alpha \big/ \sim$ (where identify for any $x \in U_\alpha \cap U_\beta$, $x \in U_\alpha$ with $x \in U_\beta$)

$$\bigcup U_\alpha \big/ \sim \;\cong\; B.$$

exercise: This is a fiber bundle.

In fact, up to a suitable notion of equivalence of such data $\{U_\alpha\}, \{\phi_{\alpha\beta}\}$, we can completely describe fiber bundles on B up to iso. this way.

(the notion of equivalence should allow one to "refine" a cover)

Key example: $S^n =$ 

$S^n_+ \cap S^n_- \underset{\text{h.e.}}{\simeq} S^{n-1}$ (thought of as equator)

Since each $S^n_{+/-}$ is contractible, any bundle on $S^n_{+/-}$ is trivial.

Given e.g., a complex vector bundle of rank k E on S^n,

have $E_+ := E|_{S^n_+} \cong S^n_+ \times \mathbb{C}^k$, and $E_- := E|_{S^n_-} \cong S^n_- \times \mathbb{C}^k$.

Using the clutching construction, can think of $E \cong E_+ \cup E_- \big/ \sim$

where along $S^n_+ \cap S^n_-$, we identify $(x, e_+) \in E_+$ with $(x, \rho(x) e_+) \in E_-$ for a clutching fcn. $\rho : S^n_+ \cap S^n_- \longrightarrow GL(k, \mathbb{C})$.

Claim: E only depends on the homotopy class of ρ $\in [S^n_+ \cap S^n_- \longrightarrow GL(k,\mathbb{C})]$

$$\cong [S^{n-1}, GL(k,\mathbb{C})] \cong \pi_{n-1}(GL(k,\mathbb{C})).$$

i.e., $\mathrm{Vect}_k^{\mathbb{C}}(S^n) \cong [S^{n-1}, GL(k,\mathbb{C})]$.

Sketch: construct a map $\Phi : \mathrm{Vect}_k^{\mathbb{C}}(S^n) \to [S^{n-1}, GL(k,\mathbb{C})]$ & check its inverse to the map $\Psi : [S^{n-1}, GL(k,\mathbb{C})] \to \mathrm{Vect}_k^{\mathbb{C}}(S^n)$

which takes $[\rho]$ to $E_+ \cup E_- \big/ (x,e_+) \sim (x, f(x) e_+)$ ← this only depends on $[f]$ up to iso? (exercise: sketch, if $f_0 \simeq f_1$, then using the homotopy, get a vector bundle over $S^n \times I$ restricting to the v.b. for f_0 & f_1 at $S^n \times \{0\}$ & $S^n \times \{1\}$ $\Rightarrow$ they're isomorphic).

Given $E \to S^n$, first we claim that $E|_{S^n_+}$ & $E|_{S^n_-}$ each admit a unique trivialization up to homotopy.
(Any two trivializations of a rank k vec. bundle on S^n_+ resp. S^n_- differ by a map $S^n_{+/-} \to GL(k,\mathbb{C})$,
but $[S^n_{+/-}, GL(k,\mathbb{C})] = \{*\}$ b/c $S^n_{+/-}$ is contractible and $GL(k,\mathbb{C})$ is connected ← (not true for $GL(k,\mathbb{R})$!)).

Using the canonical up to homotopy trivialization, define $\Phi(E)$ to be the (therefore canonical up to homotopy) clutching function associated to the trivialization.

Now, check Φ is inverse to Ψ. $\boxtimes$

— 3/10/2021 —

As an application of the above, can classify complex line bundles $(k=1)$ on n-spheres: the clutching construction says that

$$\mathrm{Vect}_1^{\mathbb{C}}(S^n) = [S^{n-1}, GL_1(\mathbb{C}) = \mathbb{C}\setminus 0] = \begin{cases} \{*\} & n=1 \\ \mathbb{Z} & n=2. \\ \{*\} & n>2. \end{cases}$$

← b/c $GL_1(\mathbb{C})$ path-connected

$[S^1, \mathbb{C}\setminus 0] \cong [S^1, S^1] \underset{\text{degree}}{\cong} \mathbb{Z}$.

↑ b/c [simply connected, $\mathbb{C}\setminus 0 \underset{\text{h.e.}}{\simeq} S^1$]
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$\{*\}$.

(Eventually, we'll see that $\mathrm{Vect}_1^{\mathbb{C}}(X)$ has a group structure by $\otimes$, and
$\mathrm{Vect}_1^{\mathbb{C}}(S^2) \cong \mathbb{Z}$ as groups.) .

main point: any line bundle $\mathcal{L}$ has an inverse with respect to $\otimes$, namely $\mathcal{L}^*$.

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Rmk: If $\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr] & & E' \arrow[dl] \\ & X & \end{tikzcd}$ map of vector bundles which is a fiberwise isomorphism, then φ is automatically a homeomorphism.
(exercise: point is that φ^{-1} automatically continuous, eventually this follows from the fact
$A \mapsto A^{-1}: GL(k) \circlearrowleft$ is continuous).

—

Classifying spaces for vector bundles (w/ remarks about classifying spaces for principal bundles).

Recall: introduced $G_k(\mathbb{R}^N)$ Grassmannian of k-planes in $\mathbb{R}^N$ (similarly $G_k(\mathbb{C}^N)$), along with
$E_{taut} \to G_k(\mathbb{R}^N)$ $(E_{taut} \to G_k(\mathbb{C}^N))$ rank k tautological bundle (cplx. rank in $\mathbb{C}$ case)

Let $\mathbb{R}^\infty = \bigcup_{N \geq 0} \mathbb{R}^N$ (thinking of $\mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \cdots$) w/ weak limit topology
$\vec{x} \longmapsto (\vec{x}, 0)$
(meaning $A \subset \mathbb{R}^\infty$ is closed iff $A \cap \mathbb{R}^N$ $\forall N$), and define

$G_k(\mathbb{R}^\infty) := \bigcup_{N \geq 0} G_k(\mathbb{R}^N)$ (note $G_k(\mathbb{R}^1) \hookrightarrow G_k(\mathbb{R}^2) \hookrightarrow \cdots$). This again comes

↑ $\emptyset$ if $1<k$.

a tautological bundle $E_{taut} \to G_k(\mathbb{R}^\infty)$, of rank k.

Similarly have $E_{taut} \to G_k(\mathbb{C}^\infty)$.

These are the "universal" rank k (real or complex) rank k vector bundles. More precisely, we have the following in the $\mathbb{R}$ case, & completely analogous statement in $\mathbb{C}$ case:

Theorem: X paracompact (e.g., a CW complex). Then:

called the 'classifying map' for E.

(1) For any rank k vector bundle $E \xrightarrow{\pi} X$, $E = f^* E_{taut}$ for some map $f : X \to G_k(\mathbb{R}^\infty)$.

(2) If we have $f_0, f_1 : X \to G_k(\mathbb{R}^\infty)$ with $f_0^* E_{taut} \cong E \cong f_1^* E_{taut}$ then $f_0 \simeq f_1$

(i.e., the classifying map f in (1) is unique up to homotopy).

In other words, the map

$$\begin{tikzcd} {[X, G_k(\mathbb{R}^\infty)]} \arrow[r, "\overset{(\text{Thm})}{\cong}"] & \mathrm{Vect}_k^{\mathbb{R}}(X) \\ {[f]} \arrow[r, mapsto] & {[f^* E_{taut}]} \end{tikzcd}$$

is an isomorphism.

---

e.g., Euclidean metric on $\mathbb{R}^\infty$

Rmk: By considering the $GL(k)$ bundle $\begin{tikzcd} \mathrm{Frame}(E_{taut}) \arrow[d] \\ Gr_k(\mathbb{R}^\infty) \end{tikzcd}$ or the $O(k)$-bundle $\begin{tikzcd} \mathrm{OFrame}(E_{taut}, \langle -,- \rangle) \arrow[d] \\ Gr_k(\mathbb{R}^\infty) \end{tikzcd}$,

the theorem also implies

$$[X, Gr_k(\mathbb{R}^\infty)] \xrightarrow{\cong} \mathrm{Bun}_{GL(k,\mathbb{R})}(X) \xrightarrow{\cong} \mathrm{Bun}_{O(k)}(X)$$

(iso. b/c $O(k) \hookrightarrow GL(k,\mathbb{R})$ is a homotopy equivalence, On the vec. bundle side, this is manifested by the fact that while a vec. bdle may admit more than one $\langle -,- \rangle$, there is a contractible space of $\langle -,- \rangle$'s; hence unique up to htpy equivalence).

Q: is there an analogous result for other $\mathrm{Bun}_G(X)$'s, G another group?

Yes:

Thm: (Milnor): G any top. group. there exists a classifying space for G-bundles (unique up to weak homotopy equivalence), meaning a space BG & a G-bundle $\begin{tikzcd} EG \arrow[d] \\ BG \end{tikzcd}$, such that the map

"classifying space of G."

"universal G-bundle"

$$\begin{tikzcd} {[X, BG]} \arrow[r, "\cong"] & \mathrm{Bun}_G(X) \\ {[f]} \arrow[r, mapsto] & {[f^* EG]} \end{tikzcd}$$

is an iso.

The pair (BG, EG) is characterized by (weak) contractibility of EG.

unitary group.

In light of above, we often simply call $G_k(\mathbb{R}^\infty) =: BO(k)$, & $G_k(\mathbb{C}^\infty) =: BU(k)$.

Example applications of Thm:

- real line bundles: Thm says $\mathrm{Vect}_1^{\mathbb{R}}(X) \cong [X, \mathbb{R}P^\infty]$
  - if $X = S^1$, know $[S^1, \mathbb{R}P^\infty] \cong \pi_1(\mathbb{R}P^\infty) = \mathbb{Z}/2$. Indeed, up to equiv. there are two real line bundles on S^1, trivial bundle, and Mobius bundle.

- if $X = S^n$, $[S^n, \mathbb{RP}^\infty] = \{*\}$.
  $n > 1$ (b/c maps lift to universal cover S^∞, which is contractible).

- complex line bundles are similarly classified by $[X, \mathbb{CP}^\infty]$

  e.g., $[S^2, \mathbb{CP}^\infty] \cong \mathrm{Vect}_1^{\mathbb{C}}(S^2) = \mathbb{Z}$. by clutching.

  & $[S^n, \mathbb{CP}^\infty] = \{*\}$ for $n \neq 2$ (also by clutching).

(basically $\Longrightarrow \pi_k(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & k=2 \\ \{*\} & \text{else} \end{cases}$, as sets at least.)

Pf of theorem:

Let $E \xrightarrow{\pi} X$ be as in theorem statement. Fix a cover $\{U_\alpha\}$ of X over which E is trivial, along w/ trivializations $\phi_\alpha : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{R}^k$.

$$\begin{tikzcd} E|_{U_\alpha} \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^k \arrow[dl, "\pi_{U_\alpha}"] \\ & U_\alpha & \end{tikzcd}$$

Define $\eta_\alpha := \pi_{\mathbb{R}^k} \circ \phi_\alpha : E|_{U_\alpha} \longrightarrow \mathbb{R}^k$.
Note: $(\eta_\alpha)|_{E_x} : E_x \xrightarrow{\cong} \mathbb{R}^k$ for each $x \in U_\alpha$

By paracompactness, we can WLOG assume U_α countable + locally finite, & can pick a subordinate partition of unity $\{f_\alpha : X \to \mathbb{R}\}$ to $\{U_\alpha\}$

(means: $f_\alpha : X \to [0,1]$ continuous, $\mathrm{supp}(f_\alpha) \subset U_\alpha$, and $\sum f_\alpha \equiv 1$).

well-defined b/c finite sum of non-zero #'s at each $p \in X$ (by loc. finiteness of $\{U_\alpha\}$)

Consider $f_\alpha \eta_\alpha : E \longrightarrow \mathbb{R}^k$, a map which is linear on each fiber of E. Summing these together gives:

$$(\ast) \quad \Phi := \bigoplus_\alpha f_\alpha \eta_\alpha : E \longrightarrow \bigoplus_\alpha \mathbb{R}^k = \mathbb{R}^\infty$$

countable sum

This map is continuous, linear on each fiber $E_x \subset E$, and injective on each fiber $E_x \subset E$.

(exercise)

(given $x \in X$, some $f_\beta(x) \neq 0$ and hence $f_\beta \eta_\beta : E_x \xrightarrow{\cong} \mathbb{R}^k$, so Φ is injective on E_x).

Then define

$$\begin{aligned} f : X &\longrightarrow G_k(\mathbb{R}^\infty) \\ x &\longmapsto \Phi(E_x) \end{aligned}$$

this is a k-dim'l subspace, hence gives a point in $G_k(\mathbb{R}^\infty)$, by injectivity above.

f classifies E? Observe there's a natural vector bundle map

b/c $\begin{smallmatrix} E_{taut} \\ \downarrow \\ Gr_k(\mathbb{R}^\infty) \end{smallmatrix}$ is a sub-bundle of $\begin{smallmatrix} Gr_k(\mathbb{R}^\infty) \times \mathbb{R}^\infty \\ \downarrow \\ Gr_k(\mathbb{R}^\infty) \end{smallmatrix}$.

$$\begin{tikzcd} E \arrow[rr, "\Psi"] \arrow[dr] & & f^* E_{taut} \subset X \times \mathbb{R}^\infty \arrow[dl] \\ & X & \end{tikzcd}$$

given by $\Psi(e) := (\pi(e), \Phi(e)) \subset X \times \mathbb{R}^\infty$.

as in $\ast$

(check: lands in $f^* E_{taut}$).

Injective on each fiber: $\Rightarrow$ Ψ induces $E \xrightarrow{\cong} f^* E_{taut}$. (note: we used Rmk that says that such a Ψ is automatically a homeomorphism). This establishes (1).

(2) Say we have $f_0, f_1 : X \longrightarrow G_k(\mathbb{R}^\infty)$ with $f_0^* E_{taut} \cong E \cong f_1^* E_{taut}$.

Let ψ_i:
$$\begin{tikzcd} E \arrow[rr, "\cong"] \arrow[dr] & & f_i^* E_{taut} \arrow[dl] \\ & X & \end{tikzcd}$$
for $i = 0, 1$.

Again we'll think of ψ_i as coming from a (linear $\overset{\text{inj.}}{\text{in}}$ each fiber) map to $\mathbb{R}^\infty$ as follows:

For each $x \in X$, $(\psi_i)_x : E_x \rightarrow (f_i^* E_{taut})_x = (E_{taut})_{f_i(x)} = f_i(x) \underset{\text{subspace}}{\subset} \mathbb{R}^\infty$.

Hence ψ_i induces $\Psi_i : E \longrightarrow \mathbb{R}^\infty$ (w/ $\Psi_i|_{E_x} = (\psi_i)_x : E_x \longrightarrow \mathbb{R}^\infty$ as above)
linear and injective on each fiber, for $i = 0, 1$.

(Note that Ψ_i determines f_i also by $f_i(x) := \Psi_i(E_x) \in Gr_k(\mathbb{R}^\infty)$, $i = 0, 1$).

Special case: Suppose for each $\overset{\text{non-zero}}{\wedge}$ $e \in E$, $\Phi_0(e)$ is not a negative multiple of $\Phi_1(e)$. $(\star)$

Then, if we set

$$\Phi_t(e) = (1-t)\Phi_0(e) + t\,\Phi_1(e) \quad \text{for } t \in [0,1], \text{ and note}$$

$\Phi_t : E \rightarrow \mathbb{R}^\infty$ continues to be injective on each fiber, so this gives

$$\begin{aligned} f_t : X &\longrightarrow G_k(\mathbb{R}^\infty), \quad \text{a homotopy } f_0 \simeq f_1 \\ x &\longmapsto \Phi_t(E_x) \end{aligned}$$

General case:

Observe that we have the ∞-codimension subspace maps

$$\begin{aligned} F_{odd} : \mathbb{R}^\infty &\longrightarrow \mathbb{R}^\infty \\ (x_1, x_2, x_3, \ldots) &\longmapsto (x_1, 0, x_2, 0, x_3, 0, \ldots) \end{aligned}$$

$$\begin{aligned} F_{even} : \mathbb{R}^\infty &\longrightarrow \mathbb{R}^\infty \\ (x_1, x_2, x_3, \ldots) &\longmapsto (0, x_1, 0, x_2, 0, x_3, \ldots) \end{aligned}$$

and moreover $(F_{odd})_s : (1-s)\,Id_{\mathbb{R}^\infty} + s\,F_{odd}$ remain injective for each $s \in [0,1]$.
$(F_{even})_s : (1-s)\,Id_{\mathbb{R}^\infty} + s\,F_{even}$ (including $s=1$)

So F_{odd}, F_{even} induce

$\widehat{F}_{odd}, \widehat{F}_{even} : G_k(\mathbb{R}^\infty) \circlearrowleft$ with $\widehat{F}_{odd} \underset{\text{by } \widehat{(F_{odd})}_s}{\simeq} id \underset{\text{by } \widehat{(F_{even})}_s}{\simeq} \widehat{F}_{even}$.

Now, given general $f_0, f_1 : X \longrightarrow G_k(\mathbb{R}^\infty)$ & Ψ_0 and $\Psi_1 : E \longrightarrow \mathbb{R}^\infty$ as above, replace Ψ_0 by the homotpic $F_{odd} \circ \Psi_0$ and Ψ_1 by homotpic $F_{even} \circ \Psi_1$.

$\Rightarrow$ replaces f_0 by homotpic $\widehat{F}_{odd} \circ f_0$ and f_1 by $\widehat{F}_{even} \circ f_1$.

Now since $\underset{\text{of the form } (x_1, 0, x_2, 0, \text{—})}{F_{odd} \circ \Psi_0(\underset{\text{non-zero}}{e})}$ cannot be a negatie multiple of $\underset{\text{of the form } (0, x_1, 0, x_2, \text{— —})}{F_{even} \circ \Psi_1(e)}$, $\overset{\text{i.e., satisfies } (✪)}{\ }$ we've reduced to special case. $\blacksquare$.