Characteristic classes

A **characteristic class** for real or complex vector bundles (or for real/cplx. vec. bundles of rank k) assigns to each such $E \to B$ a coh. class $c(E) \in H^*(B;R)$ ← some R, may depend on c.

(only depends on iso. class of E &) which is natural in E in the sense that if $f: A \to B$ continuous map, we get a pullback bundle $\begin{array}{c} f^*E \\ \downarrow \\ A \end{array}$, and

$$\underbrace{c(f^*E)}_{\in H^\bullet(A;R)} = \overset{\uparrow}{f^*}\big(\underbrace{c(E)}_{H^\bullet(B;R)}\big).$$

$f^*: H^\bullet(B;R) \to H^\bullet(A;R)$.

By the existence of classifying maps for vector bundles, such a class c is determined on all $E \to B$ by knowing

- (if complex rank k bundles) $\hat{c} := c\left(\begin{array}{c} E^{k,\mathbb{C}}_{\text{taut}} \\ \downarrow \\ BU(k) \end{array}\right) \in H^\bullet(BU(k);R) = H^\bullet(G_k(\mathbb{C}^\infty);R)$.

  (for any other $\begin{array}{c} E \\ \downarrow \\ B \end{array}$, $E = f^* E_{\text{taut}}$ for some $f: B \to BU(k)$ unique up to homotopy, so naturality forces $c(E) := f^*\hat{c}$.)

- (if real rank k bundles) $\hat{c} := c\left(\begin{array}{c} E^{k,\mathbb{R}} \\ \downarrow \\ BO(k) \end{array}\right) \in H^\bullet(BO(k);R) = H^\bullet(G_k(\mathbb{R}^\infty);R)$.

Obs: If $E \to B$ is trivial, then $E \cong p^*\underline{\mathbb{R}}^k$ (or $p^*\underline{\mathbb{C}}^k$ if cplx. case) where $p: B \to pt$.

$$\Rightarrow c(E) = p^*\Big(\underbrace{c\Big(\begin{array}{c} \underline{\mathbb{R}}^k \\ \downarrow \\ pt. \end{array}\Big)}_{\in H^\bullet(pt.) = \begin{cases} R & \deg 0 \\ 0 & \text{otherwise.} \end{cases}}\Big)$$

is trivial, in sense that it's either 0 or a non-zero multiple of unit in H^0.

We conclude if $c(E)$ is not trivial in such a sense (i.e., non-zero in some degree >0), then E cannot be a trivial bundle.

First examples:

(1) the <u>first Stiefel-Whitney class</u> of a real line bundle $L \to X$ (gives a class $w_1(L) \in H^1(X;\mathbb{Z}/2)$):

In $BO(1) = G_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$, there exists a unique non-zero element $h \in H^1(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

(as a ring $H^\bullet(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[h]$, $|h| = 1$).

Define $w_1(L_{\text{taut}} \to \mathbb{RP}^\infty) := h$.

$\Rightarrow$ for any $L \to X$ classified by $X \xrightarrow{f} \mathbb{RP}^\infty$ (i.e., $L = f^*L_{\text{taut}}$), we get a def'n

$$\underset{\substack{\uparrow \\ \text{well-defined b/c } f \text{ well-defined up to homotopy}}}{w_1(L)} := f^*(h) \in H^1(X;\mathbb{Z}/2) \underset{\substack{\uparrow \\ \text{no torsion in } H_0}}{\xrightarrow{UCT}} \mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z}/2) \underset{\substack{\uparrow \\ \text{b/c } \pi_1(X)^{ab} = H_1(X)}}{=} \mathrm{Hom}(\pi_1(X),\mathbb{Z}/2)$$

Given a loop $\gamma: S^1 \to X$, $w_1(L)([\gamma]) \in \mathbb{Z}/2$ is defined as $\begin{cases} 1 & \text{if } \gamma^*L \to S^1 \text{ is non-trivial} \\ 0 & \text{if } \gamma^*L \to S^1 \text{ is trivial.} \end{cases}$

(2) The first Chern class of a complex line bundle $L \longrightarrow X$ (gives a class $c_1(L) \in H^2(X;\mathbb{Z})$)):

In $BU(1) = G_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty$, note $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[h]$ with $|h|=2$ and in particular $H^2(\mathbb{CP}^\infty) \cong \mathbb{Z}$.

We want to declare $c_1(L^{\mathbb{C}}_{taut})$ = a generator of $H^2(\mathbb{CP}^\infty)$, but which one? (two choices, so far h is only defined as a choice of generator of H^2). The choice is a convention, but we need to fix one.

We'll use the following facts to fix an iso. $H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}$.

- a complex vector space $V/_{\mathbb{C}}$ of finite dimension has a canonical orientation when thought of as a real vector space:
  Namely if $v_1, \dots, v_n$ is a basis over $\mathbb{C}$ declare "complex-orientation" of $V/_{\mathbb{R}}$ to be orientation induced by $(v_1, iv_1, v_2, iv_2, \dots, v_n, iv_n)$.
  obs: if swap v_s & v_t, in real basis above need to swap (v_s, iv_s) w/ (v_t, iv_t) ⇝ even # of swaps ⇝ same orientation.
  - More generally, since $GL(n,\mathbb{C})$ is connected, the image $GL(n,\mathbb{C}) \hookrightarrow GL(2n,\mathbb{R})$ lands in a connected component of $GL(2n;\mathbb{R})$, i.e., $GL(2n,\mathbb{R})^+$. (b/c it contains Id).

- In particular, complex manifolds M carry canonical orientations of their tangent bundle $\begin{matrix} TM \\ \downarrow \\ M \end{matrix}$ (thought of as a real bundle). — pick the complex orientation for every T_pM; canonical.

- In particular, for a cpt. complex manifold Q^{2n} (real dim. $2n$) using equivalence between homology orientations & orientations of $\begin{matrix} TM \\ \downarrow \\ M \end{matrix}$ (omitted, but proved in many places), we deduce $\exists$ a canonical fundamental class.
  $[Q] \in H_{2n}(Q;\mathbb{Z})$.

- So $\exists$ a canonical $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^1;\mathbb{Z})$ & $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$, a canonical generator $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^\infty;\mathbb{Z})$

- Define $h \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ to be the generator with $\langle h, [\mathbb{CP}^1] \rangle = +1$.

Declare $c_1\left(\begin{matrix} L_{taut} \\ \downarrow \\ \mathbb{CP}^\infty \end{matrix}\right) := -h$ where h is the canonical generator above.

⇒ gives a def'n for any $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ classified by $f: X \to \mathbb{CP}^\infty$ (so $f^*L_{taut} \cong L$), as:

$$c_1(L) := f^*(-h) \in H^2(X;\mathbb{Z}).$$

Lemma: $L_1, L_2 \longrightarrow X$ cplx. line bundles, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in H^2(X;\mathbb{Z})$
(and same lemma holds for w_1 in case of real line bundles w/ same proof; replace $\mathbb{CP}^\infty$ by $\mathbb{RP}^\infty$, etc.)

Pf: Say $f_i : X \to \mathbb{C}P^\infty$ classifies L_i (so $f_i^* L_{taut} = L_i$) $i = 1, 2$.

& define $F = (f_1, f_2) : X \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$.

Let $\pi_i : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ projection to ith factor, $i = 1, 2$, &

set $L_i^{taut} := \pi_i^* L_{taut} \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$

(Rmk: For any $\begin{matrix} E \\ \downarrow \\ A \end{matrix}$, $\begin{matrix} F \\ \downarrow \\ B \end{matrix}$, $\begin{matrix} E \boxtimes F := (\pi_A^* E) \otimes (\pi_B^* F) \\ \downarrow \\ A \times B \end{matrix}$)

Obs: $L_1 \otimes L_2 = F^*(\overbrace{L_1^{taut} \otimes L_2^{taut}}^{\text{"}L_{taut} \boxtimes L_{taut}\text{"}})$

(why? $F^*(L_1^{taut} \otimes L_2^{taut}) = F^*(L_1^{taut}) \otimes F^*(L_2^{taut})$

$= ((f_1, f_2)^* \pi_1^* L_{taut}) \otimes ((f_1, f_2)^* \pi_2^* L_{taut})$

$= (f_1^* L_{taut}) \otimes (f_2^* L_{taut})$

$= L_1 \otimes L_2$.).

In $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$, we know $H^\bullet(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; \mathbb{Z}) \underset{\text{Künneth}}{\cong} \mathbb{Z}[h_1, h_2]$, $|h_1| = |h_2| = 2$

$h_1 := \pi_1^* h$, $h_2 := \pi_2^* h$, h canonical element as above.

which in degree 2 is $\mathbb{Z}\langle h_1 \rangle \oplus \mathbb{Z}\langle h_2 \rangle$.

Claim: $c_1(L_1^{taut} \otimes L_2^{taut}) = -h_1 - h_2$.

If true, then by Obs: $c_1(L_1 \otimes L_2) = c_1(F^*(L_1^{taut} \otimes L_2^{taut})) = F^*(-h_1 - h_2)$

$= (f_1, f_2)^*(\pi_1^*(-h) + \pi_2^*(-h)) = f_1^*(-h) + f_2^*(-h) = c_1(L_1) + c_1(L_2)$.

so we'd be done.

Pf of claim: know $c_1(L_1^{taut} \otimes L_2^{taut}) = a h_1 + b h_2$; need to pin down a & b.

restricting along $\mathbb{C}P^\infty \times pt \overset{i_1}{\hookrightarrow} \mathbb{C}P^\infty \times \mathbb{C}P^\infty$:

$i_1^* L_2^{taut} \cong \underline{\mathbb{C}}$ & $i_1^* L_1^{taut} = L_{taut}$, so $i_1^*(L_1^{taut} \otimes L_2^{taut}) \cong L_{taut}$,

and
$$\begin{aligned} i_1^* : H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) &\longrightarrow H^2(\mathbb{C}P^\infty). \\ h_1 &\longmapsto h \\ h_2 &\longmapsto 0. \end{aligned}$$

so
$$\begin{tikzcd} i_1^* c_1(L_1^{taut} \otimes L_2^{taut}) = i_1^*(a h_1 + b h_2) = a h \\ \\ c_1(i_1^*(L_1^{taut} \otimes L_2^{taut})) = c_1(L_{taut}) = -h. \arrow[uu, phantom, "\parallel"] \end{tikzcd}$$

$\updownarrow$ $a = -1$.

Similarly, restricting along $pt \times \mathbb{C}P^\infty \xrightarrow{i_2} \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ & comparing $\Rightarrow b = -1$ as desired. ☒.

Rmk: {complex line bundles X, $\otimes$} form a group (identity element: $\underline{\mathbb{C}}$, and inverse of L is $L^* := \mathrm{Hom}_{\mathbb{C}}(L, \underline{\mathbb{C}})$. exercise: verify that $L^* \otimes_{\mathbb{C}} L \cong \underline{\mathbb{C}}$).

(similarly for real line bundles)

So: c_1: {$\overset{\text{cplx}}{\text{line}}$ bundles, $\otimes$} $\longrightarrow H^2(X;\mathbb{Z})$ is a group homomorphism.

In fact: c_1 induces an isomorphism $(\mathrm{Vect}^1_{\mathbb{C}}(X), \otimes) \xrightarrow{\cong} H^2(X;\mathbb{Z})$, complete invariant!

We won't prove this right now, one way to see it is to understand that $\mathbb{CP}^\infty = BU(1)$ is the Eilenberg-Maclane space $K(\mathbb{Z},2)$; maps

$$[X, \mathbb{CP}^\infty = BU(1) = K(\mathbb{Z},2)] \xrightarrow{\cong} H^2(X;\mathbb{Z})$$

$$[f] \longmapsto f^*(h).$$

(More generally, $\exists$ $K(A,n)$, & classes $\alpha \in H^n(K(A;n);A)$,

s.t. $[X, K(A,n)] \xrightarrow{\cong} H^n(X;A)$

$[f] \longmapsto f^*\alpha$.)

(nice paper topic!)

Higher Chern and Stiefel-Whitney classes in general

for cplx. vec. bundles; for real vec. bundles

There is a completely axiomatic characterization of Chern + Stiefel-Whitney classes which we now describe:

Thm: (Stiefel-Whitney classes): $\exists$ unique characteristic classes w_i of real-vector bundles, $i \geq 1$, w/ $w_i(E) \in H^i(B;\mathbb{Z}/2)$ for $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ (vec. bdle of any rank) depending only on the iso. type of E (so $w_i : \mathrm{Vect}_k^{\mathbb{R}}(B) \to H^i(B;\mathbb{Z}/2)$)

satisfying:

(a) (naturality): w_i are char. classes, i.e., $w_i(f^*E) = f^* w_i(E)$ any $f: A \to B$.

(b) (Whitney sum formula)

Denoting by $w(E) = \overset{w_0(E) \text{ by convention.}}{1} + w_1(E) + w_2(E) + \cdots \in H^*(B;\mathbb{Z}/2)$ the "total Stiefel-Whitney class";

(so part in degree i is $w_i(E)$)

then $\boxed{w(E_1 \oplus E_2) = w(E_1) \cup w(E_2).}$

(explicitly taking degree s parts of both sides:

$$w_s(E_1 \oplus E_2) = \sum_{\substack{i+j=s \\ i \geq 0 \\ j \geq 0}} w_i(E_1) \cup w_j(E_2)).$$

i.e., $w_2(E_1 \oplus E_2) = w_2(E_1) + w_1(E_1) \cup w_1(E_2) + w_2(E_2)$, etc.)

(c) (dimension) $w_i(E) = 0$ for $i > \mathrm{rank}_{\mathbb{R}}(E)$,

(d) (normalization) $w_1(L_{taut} \to \mathbb{RP}^\infty)$ is the unique generator of $H^1(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

(in fact declaring $w_1(L_{taut} \to \mathbb{RP}^1) \neq 0$ is sufficient — exercise to see this determines (d).).

Thm: (Chern classes) : $\exists$ unique characteristic classes c_i of cplx vector bundles, $i \geq 1$, w/ $c_i(E) \in H^{2i}(B; \mathbb{Z})$ for $\begin{array}{c}E\\ \downarrow\\ B\end{array}$ ← (vec. bdle of any rank) depending only on the iso. type of E (so $w_i : \underset{\text{any } k}{Vect_k^{\mathbb{C}}(B)} \to H^{2i}(B;\mathbb{Z})$)

satisfying:

(a) (naturality) : c_i are char. classes, i.e., $c_i(f^*E) = f^* c_i(E)$ any $f: A \to B$.

(b) (Whitney sum formula)

Denoting by $c(E) = \overset{c_0(E) \text{ by convention}}{1} + c_1(E) + c_2(E) + \text{——} \in H^*(B; \mathbb{Z})$ the "total Chern class";
(so part in degree $2i$ is $c_i(E)$)

then $\boxed{c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).}$

(as above can extract out explicit formulae for each $c_s(E_1 \oplus E_2)$)

(c) (dimension) $c_i(E) = 0$ for $i > \text{rank}_{\mathbb{C}}(E)$.

(d) (normalization) $c_1(L_{taut} \to \mathbb{CP}^\infty)$ is the generator $-h \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$ where h is the canonical element specified above.

(as before, it would have sufficed to fix $c_1(L_{taut} \to \mathbb{CP}^1)$).

Next time, we'll approach construction of Chern + Stiefel Whitney classes

(many constructions in literature, we'll appeal to the Leray-Hirsch theorem ← will take some time., a tool for understanding cohomology of fiber bundles $F \to \begin{array}{c}P\\ \downarrow\\ B\end{array}$ in some circumstances; applied to $\begin{array}{c}\mathbb{P}(E)\\ \downarrow\\ B\end{array}$ ← "real or cplx. fiberwise projectivization" of E)

An observation:

- by naturality $c_i(\underline{\mathbb{C}}^k) = 0$ for any $k, i > 0$ (resp. $w_i(\underline{\mathbb{R}}^k) = 0, i > 0$)

so $c(E \oplus \underline{\mathbb{C}}^k) = c(E) \cup c(\underline{\mathbb{C}}^k) = c(E) \cdot (\underset{\text{"}c_0(\underline{\mathbb{C}}^k)\text{"}}{1}) = c(E).$

i.e., $c_j(E \oplus \underline{\mathbb{C}}^k) = c_j(E).$

& similarly $w_j(E \oplus \underline{\mathbb{R}}^k) = w_j(E).$

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3/17/2021

The Leray-Hirsch theorem is a tool for understanding coh. of total spaces of certain fiber bundles. satisfying some hypotheses

Recall that if
$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$
is a fiber bundle, then $\pi^*: H^\bullet(B;R) \to H^\bullet(E;R)$ is a ring map, equips $H^\bullet(E;R)$ w/ structure of a $H^\bullet(B;R)$-module ($b \in H^\bullet(B;R)$ acts by $b \cdot \overset{H^\bullet(E;R)}{x} := \pi^*(b) \smile x$).

Thm: (Leray-Hirsch theorem): Say $F \xrightarrow{i} E \xrightarrow{\pi} B$ a fiber bundle, & R ring s.t.

(a) $H^k(F;R)$ free & finitely generated over R for each k.

(b) The restriction map $i^*: H^\bullet(E;R) \to H^\bullet(F;R)$ is surjective.

Under the hypotheses of (a)+(b), we can choose a splitting $c: H^\bullet(F;R) \to H^\bullet(E;R)$ (not induced by a map of spaces), i.e., for any basis $\{\gamma_j \in H^{k_j}(F;R)\}_j$ of $H^\bullet(F)$ as an R-module

we obtain classes $c_j := c(\gamma_j) \in H^{k_j}(E;R)$ which restrict to the given basis $\{\gamma_j\}$. Call such a collection $\{c_j\}$ (or the map c) a cohomology extension of the fiber.

Then, the map
$$\Phi: H^\bullet(B;R) \otimes_R H^\bullet(F;R) \longrightarrow H^\bullet(E;R)$$
$$\sum b_i \otimes \gamma_j \longmapsto \sum \pi^*(b_i) \smile c_j$$

depends on the choice of coh. extension of fiber.

"$b_i \cdot c_j$" in terms of module action of $H^\bullet(B)$ on $H^\bullet(E)$.

is an isomorphism (as $H^\bullet(B;R)$-modules).

In other words, any $c \in H^\bullet(E;R)$ can be written uniquely as $\sum \pi^*(a_j) \smile c_j$ for some unique $a_j \in H^\bullet(B;R)$.

Rmks/examples:

- For a trivial fiber bundle $E = B \times F$, w/ $H^\bullet(F;R)$ free & finitely generated, have $E \xrightarrow{\pi_F} F$, & the image of $\pi_F^*: H^\bullet(F) \to H^\bullet(E)$ gives a splitting of $i^*: H^\bullet(E) \to H^\bullet(F)$. Hypotheses therefore apply, & can use $c_j := \pi_F^*(\gamma_j)$ for a given basis $\{\gamma_j\}$ of $H^\bullet(F)$. L-H for these particular c_j's is just Künneth. (Künneth: any $c \in H^\bullet(B \times F)$ can be written as $\sum \pi^*(\underset{H^\bullet(B)}{a_j}) \smile \pi^*(\underset{H^\bullet(F)}{\gamma_j})$)

  L-H is more general in a sense, as it allows other choices of c_j (but this can also be extracted from Künneth).

- unlike Künneth, L-H theorem does not assert that $H^\bullet(E) \cong H^\bullet(B) \otimes H^\bullet(F)$ as rings! This can be false. (all one gets is that $H^\bullet(B) \otimes H^\bullet(F) \cong H^\bullet(E)$ as $H^\bullet(B)$-modules).

  (alg. example: $S = k[\underset{\deg 3}{x}]/x^5$, $T = k[\underset{\deg 2}{y}]/y^2$, now there's an iso. of S-modules
  $$k[x,y]/_{x^5, y^2} \cong S \otimes T \cong k[\overset{\deg 3}{x}, \overset{\deg 2}{y}]/_{x^5,\, y^2-1}$$
  but not as rings!)
  $$x \longmapsto x$$
  $$y \longmapsto y.$$

- Example where L-H theorem fails to apply:

Look at the Hopf bundle
$$\begin{tikzcd} \overset{F=}{S^1} \arrow[r] & S^3 = E \arrow[d] \\ & S^2 = B \end{tikzcd}$$
. Then $H^\bullet(S^3)$ cannot surject on $H^\bullet(S^1)$ as a graded R-module, b/c $H^1(S^1) \cong R$, but $H^1(S^3) = 0$.

<u>Proof of the Leray-Hirsch theorem</u>, <u>detailed sketch</u>:

Steps: (1) Prove for finite dimensional CW complexes $B = B^n$ ← meaning, prove theorem for all $\begin{matrix} F \to E \\ \downarrow \\ B \end{matrix}$ satisfying hypotheses where B is finite-dim'l CW.

(2) Prove for all CW complexes $B = \bigcup_{n \geq 0} B^n$ ← a little sketchy.

(3) Prove for all spaces by "CW-approximation" theorem. ← sketchiest part.

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(1) For finite dim'l CW complexes, we'll induct on $\dim(B)$.

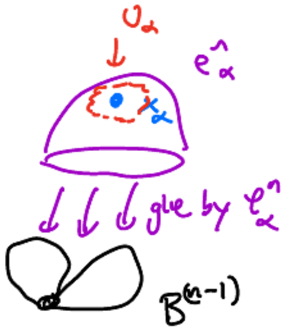
- true when B is 0-dim'l (b/c in this case $E = \coprod_{x \in B^0} \{F_x\}$)

In this case $H^\bullet(B) = H^0(B^0) = \prod_{x \in B^0} \mathbb{Z}\langle 1_x \rangle$, and $H^k(E) = \prod_{x \in B^0} H^k(F_x) \underset{\text{check}}{\cong} H^k(F) \otimes H^0(B^0)$

(exercise: spell out details)

- Say it's true for all $(n-1)$-dim'l CW complexes, and let

$$B = B^{(n-1)} \cup \bigcup_{\alpha \in A} e_n^\alpha \quad (\text{along } \varphi_\alpha^n : \partial e_n^\alpha \to B^{(n-1)}).$$



Have $F \to E \to B$ satisfying hypotheses of L-H.

- Pick $x_\alpha \in \operatorname{int}(e_\alpha^n)$ for each α, and let $\mathring{e}_n^\alpha := e_n^\alpha \setminus x_\alpha$.

Let $B' := B^{(n-1)} \cup \bigcup \mathring{e}_n^\alpha \subseteq B$, and denote by $E|_{B'} =: E'$

<u>First observation</u>: B' deformation retracts to $B^{(n-1)}$ (by retracting each $\mathring{e}_n^\alpha$ to ∂e_n^α),

and we want to similarly deduce that $E|_{B^{(n-1)}} \underset{\substack{\text{homotopy} \\ \text{equiv.}}}{\simeq} E|_{B'}$ (hence induces iso. on coh. groups)

apply below Lemma to $X = B'$, $X' = B^{(n-1)}$:

<u>Lem</u>: Give $\pi : P \to X$ (X paracompact) fiber bundle, say X def. retracts to $X' \subset X$. Then $P|_{X'} \subset P|_X$ is a homotopy equivalence.

Pf <u>sketch</u>: Let $f_t : X \to X'$ be the def. retraction, i.e., $f_0 = \mathrm{id}_X$, $f_1(X) \subset X'$, $f_t|_{X'} = \mathrm{id}_{X'}$.

Look at

$$\begin{tikzcd}
 & & P \arrow[d, "\pi"] \\
P \arrow[urr, dashed, "\mathrm{id}_P \text{ lifts } \pi"] \arrow[r, "\pi"'] \arrow[rr, bend right, "f_0\circ\pi \;=\; \mathrm{id}\circ\pi = \pi\,."'] & X \arrow[r, "f_0"'] & X
\end{tikzcd}$$

restrict to this

lift at $t=0$

$$\begin{tikzcd}
 & & P \arrow[d, "\pi"] \\
P \arrow[urr, dashed, "\text{rel HLP: } \exists h_t"] \arrow[r, "\pi"'] \arrow[rr, bend right, "f_t\circ\pi"'] & X \arrow[r, "f_t"'] & X
\end{tikzcd}$$

restricting to this lift along P'.

restricted to X' :

$$\begin{tikzcd}
 & & P \arrow[d, "\pi"] \\
P' \arrow[urr, "\mathrm{id}|_{P'} \text{ lifts } f_t|_{X'}\circ\pi = \pi\,."] \arrow[r, "\pi"] & X' \arrow[r, "f_t|_{X'} = \mathrm{id}_{X'}"'] & X
\end{tikzcd}$$

By relative homotopy lifting property, if we denote by g_t the map $f_t\circ\pi : P \to X$, g_t admits a lift $h_t : P \longrightarrow P$ (i.e., $\pi\circ h_t = g_t = f_t\circ\pi$) agreeing w/ given lift id_P at time 0 and w/ fixed lift $\mathrm{id}_{P'}$ for all time when restricted to P'.

Check: h_t provides homotopy between id_P and $P \xrightarrow{h_1} P' \xrightarrow{\text{incl}} P$; ~~i.e.,~~ & since $h_1|_{P'} = \mathrm{id}|_{P'}$, i.e., $\begin{tikzcd} P' \arrow[r, "\text{incl}"] \arrow[rr, bend right, "\mathrm{id}"'] & P \arrow[r, "h_1"] & P \end{tikzcd}$, h_1 & incl. are htpy inverse. $\boxtimes$.

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Consider the following commutative diagram (using a fixed cohomology extension of the fibre) (R implicit) of LES's:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d, "\Phi"] & H^\bullet(B,B')\otimes H^\bullet(F) \arrow[r] \arrow[d, "\Phi\ (A)"] & H^\bullet(B)\otimes H^\bullet(F) \arrow[r] \arrow[d, "\Phi\ (?)"] & H^\bullet(B')\otimes H^\bullet(F) \arrow[r, "\delta^*"] \arrow[d, "\Phi\ (B)\ \checkmark"] & {} \arrow[d, "\Phi"] \\
\cdots \arrow[r] & H^\bullet(E,E') \arrow[r] & H^\bullet(E) \arrow[r] & H^\bullet(E') \arrow[r, "\delta^*"] & {}
\end{tikzcd}$$

relative version of same map Φ using relative cup product

$$\underset{\substack{\rotatebox{90}{$\in$}\\ \pi^*(\text{class}\\ \text{in } H^\bullet(B,B'))}}{H^\bullet(E,E')} \otimes_R \underset{\substack{\rotatebox{90}{$\in$}\\ c_j}}{H^\bullet(E)} \xrightarrow{\cup} H^\bullet(E,E')$$

(top seq. is exact b/c it was LES for pair (B,B') $\otimes$ a free module $H^\bullet(F)$) . hypotheses of L-H.

(bottom seq. is LES of (E,E')).

exercise: check it's commutative. (Φ is natural, & check compat. w/ δ^* above)

If (A) & (B) are isomorphisms, then (?) will be too, by 5 lemma.

The map (B) is an iso. by induction, because :

$$\begin{tikzcd}
H^\bullet(B^{(n-1)}) \otimes H^\bullet(F) \arrow[d, "\Phi", "\text{by induction } \cong"'] \arrow[dr, phantom, "\circlearrowright"] & H^\bullet(B') \otimes H^\bullet(F) \arrow[l, "\cong"'] \arrow[d, "\Phi"] \\
H^\bullet(E|_{B^{(n-1)}}) & H^\bullet(E') \otimes H^\bullet(F) \arrow[l, "\cong"', "\text{(lemma above)}"]
\end{tikzcd}$$

$\leftarrow$ therefore this map is an $\cong$

Suffices to check (A) is an iso. By fiber bundle property, $\exists$ open U_α in $\mathrm{int}(e^n_\alpha)$ of x_α along which $E|_{U_\alpha} \cong F \times U_\alpha$ a trivial fiber bundle.

Let $U = \coprod_\alpha U_\alpha$, and let $U' = U \cap B'$ (i.e., $U' = U - \bigcup x_\alpha$.). so $E|_U \cong F \times U$.

Excision $\Rightarrow$ $H^\bullet(B,B') \cong H^\bullet(U,U') \left(\cong H^\bullet(\amalg U_\alpha, \amalg \{U_\alpha - x_\alpha\}) \right)$

and $H^\bullet(E,E') \cong H^\bullet(E|_U, E|_{U'}) \cong H^\bullet(U \times F, U' \times F)$,

Thus, (A) reduces to showing that

$$\Phi : H^\bullet(U,U') \otimes_R H^\bullet(F) \longrightarrow H^\bullet(U \times F, U' \times F) \text{ is an iso.}$$

using LES of the pair (U,U') & 5 Lemma, it suffices to show for any V, the map $\Phi : H^\bullet(V) \otimes H^\bullet(F) \longrightarrow H^\bullet(V \times F)$ is an iso. where Φ constructed using a coh. extension of fiber. (i.e., Leray-Hirsch for trivial bundles)

Exercise: Prove L-H for trivial bundles. i.e., $E = V \times F$, $H^k(F)$ free finitely gen $\forall k$ & let $c_j \in H^\bullet(E)$ be any collection of classes restricting to a basis $\{\gamma_j\}$ of $H^\bullet(F)$. Then prove that

$$H^\bullet(V) \otimes H^\bullet(F) \xrightarrow{\Phi} H^\bullet(E) \text{ is an iso.}$$
$$a \times \gamma_j \longmapsto \pi^*(a) \cup c_j .$$

(True by Künneth if one uses $\hat{c}_j = \pi_F^* \gamma_j$. Prove for a gen'l c_j by relating this choice to $\hat{c}_j$).

---

(2) General CW complex $B = \bigcup B^n$. (sketch):

we know the inclusion $B^n \subset B$ induces isos $H^i(B;R) \xrightarrow{\cong} H^i(B^n;R)$ for $i < n$.

Similarly, if $F \to E \to B$ fiber bundle,

Claim: $H^i(E;R) \xrightarrow{\cong} H^i(E|_{B^n};R)$ for $i < n$.

(this follows from the fact [a] that (B,B^n) is "n-connected"

$\Rightarrow$ (b) $(E, E|_{B^n})$ is "n-connected" too (by HLP)

$\swarrow$ means $\left(\pi_i(B,B^n) = 0 \text{ for } i \leq n. \right.$ i.e., any map $(D^i, \partial D^i) \to (B,B^n)$ is homotopic through maps $(D^i,\partial D^i) \to (B,B^n)$ to a map into $\left.(B^n, B^n). \right)$

$\Rightarrow$ (c) for any n-connected (X, X'), $H^i(X;R) \xrightarrow{\cong} H^i(X';R)$ for all $i<n$.

(by a more general property: if $f: X' \to X$ induces an iso. on all π_i for $i \leq n$, then it induces an $\cong$ on homology in degrees $i<n$ & surjection when $i=n$; similarly for cohomology by UCT. Hatcher Prop 4.21).

Using this, have

$$\begin{tikzcd}
H^\bullet(B)\otimes H^\bullet(F) \arrow[r, "\text{iso. in degree } <n"] \arrow[d, "\Phi"] & H^\bullet(B^n)\otimes H^\bullet(F) \arrow[d, "\Phi_n", "\cong"'] \\
H^\bullet(E) \arrow[r, "\text{iso in degree } <n"'] & H^\bullet(E|_{B^n})
\end{tikzcd}$$

(B^n finite CW complex).

$\Rightarrow$ For any i w/ $i<n$, we deduce Φ iso. in degree i.

But n was arbitrary, so Φ iso. in all degrees.

✓.

(3) General $F \to E \to B$. Use "CW approximation":

Thm: For any B, $\exists$ a CW cplx. A & a map $f: A \to B$ which is a "weak homotopy equivalence" (means: f induces iso. on homotopy groups).

$\Rightarrow$ $\begin{matrix} f^*E \\ \downarrow \\ A \end{matrix}$ is a fibration, and

$$\begin{tikzcd}
f^*E \arrow[r, "\sim", "\text{w.e.}"'] \arrow[d] & E \arrow[d] \\
A \arrow[r, "\sim", "\text{w.e.}"'] & B
\end{tikzcd}$$

again (why? uses "LES of a fibration in homotopy groups" + 5 Lemma)

Now, by general theory, weak equivalence induce $\cong$ on cohomology and homology. (see aside above).

In particular, $\{c_j\}$ pull back to classes in $H^\bullet(f^*E;R)$ restricting to a basis in each fibre, so naturality of Φ reduces L-H for $E \to B$ to L-H for $f^*E \to A$ (✓ by (2)):

$$\begin{tikzcd}
H^\bullet(A)\otimes H^\bullet(F) \arrow[d, "\Phi", "\text{by(2) } \cong"'] & H^\bullet(B)\otimes H^\bullet(F) \arrow[l, "\cong"'] \arrow[d, "\Phi"] \\
H^\bullet(f^*E) & H^\bullet(E) \arrow[l, "\cong"]
\end{tikzcd}$$

$\leftarrow$ therefore $\cong$.

$\boxtimes$.

—————— 3/19/2021

Construction of Chern classes, using Leray-Hirsch theorem. (+ Stiefel-Whitney classes — analogous) indicated in red

for cplx. vec. bundles

$E \xrightarrow{\pi} B$ complex rank k vector bundle (resp. real vec. bundle $E \to B$

Form $\begin{matrix} \mathbb{CP}(E) \\ \downarrow \\ B \end{matrix}$ or $\mathbb{P}(E)$ (when "$\mathbb{C}$" implicit), (analogously $\mathbb{RP}(E) \to B$, sometimes also denoted $\mathbb{P}(E)$ if $\mathbb{R}$ is implicit).

"complex fibrewise projectivization of E." This is an associated fibre bundle w/ fiber $\mathbb{P}(\mathbb{C}^k) \cong \mathbb{CP}^{k-1}$. Can construct either as $(E \setminus 0_B)/\mathbb{C}^*$ or $\mathbb{C}\text{-Frame}(E) \times_{GL(k,\mathbb{C})} \mathbb{CP}^{k-1}$.

Each fiber $\mathbb{P}(E)_b \equiv \mathbb{C}\mathbb{P}(\underset{\text{cplx. vector space}}{E_b}) \cong (E_b \setminus 0)/\mathbb{C}^*$.

There's a tautological line bundle over $\mathbb{P}(E_b)$ for each $b \in B$ as usual: $L_b^{taut} = \{(x,v) \mid x \in \mathbb{P}(E_b)$, i.e. $x \subseteq E_b$ line, and $v \in x\}$,

which assemble to give a tautological complex line bundle over $\mathbb{P}(E)$:

$$L := \{(x,v) \mid x \in \mathbb{P}(E) = \coprod_b \mathbb{P}(E_b),\ v \in x\} \longrightarrow \begin{array}{c} L \longmapsto \mathbb{P}(E) \\ (x,v) \longmapsto x. \end{array}$$

$$= \{(b,y,v) \mid b \in B,\ y \in \mathbb{P}(E_b),\ v \in y\}.$$

So, there's a class $h_p \in H^2(\mathbb{P}(E);\mathbb{Z})$ $h_p := -\underset{\substack{\uparrow \\ \text{1st chern class} \\ \text{as previously defined,}}}{c_1^{old}}(L) \overset{\text{explicitly}}{=} f^*h$, where $f: \mathbb{P}(E) \to \mathbb{C}\mathbb{P}^\infty$ classifies L so $f^*L_{taut} = L$, and $h \in H^2(\mathbb{C}\mathbb{P}^\infty;\mathbb{Z})$ canon. generator.

(in the real case, similarly have tautological real line bundle $L \to (\mathbb{R})\mathbb{P}(E)$, inducing a class $h_p = \underset{\substack{\uparrow \\ (\mathbb{Z}/2 \text{ coeffs, } +1 = -1)}}{} w_1^{old}(L) = f^*h \in H^1(\mathbb{P}(E);\mathbb{Z}/2)$, where $f: \mathbb{P}(E) \to \mathbb{R}\mathbb{P}^\infty$ classifies L, $h \in H^1(\mathbb{R}\mathbb{P}^\infty;\mathbb{Z}/2)$ non-zero element).

Now, consider $\underbrace{1 = h_p^0, h_p, h_p^2, \ldots, h_p^{k-1}}_{\text{(i.e., pullback)}} \in H^*(\mathbb{P}(E);\mathbb{Z})$. $\boxed{(\text{rank}_{\mathbb{C}}(E) = k)}$

Observe the restriction of $L \to \mathbb{P}(E)$ to a fiber $\mathbb{P}(E_b)$ is $L_{taut} \to \mathbb{P}(E_b) \cong L_{taut} \to \mathbb{C}\mathbb{P}^{k-1}$.

Therefore by naturality of c_1^{old}, h_p restricts to $-c_1^{old}(L_{taut} \to \mathbb{P}(E_b)) = h \in H^2(\mathbb{C}\mathbb{P}^{k-1};\mathbb{Z})$.

So $1, h_p, \ldots, h_p^{k-1}$ restrict to $1, h, h^2, \ldots, h^{k-1}$ the standard generators for $H^*(\mathbb{C}\mathbb{P}^{k-1};\mathbb{Z})$ as a $\mathbb{Z}$-module. (Recall as a ring $H^*(\mathbb{C}\mathbb{P}^{k-1}) \cong \mathbb{Z}[h]/h^k$, so $H^{2i}(\mathbb{C}\mathbb{P}^{k-1}) = \begin{cases} \mathbb{Z}\langle h^i \rangle & 0 \le i \le k-1 \\ 0 & \text{else.} \end{cases}$

& $H^{odd}(\mathbb{C}\mathbb{P}^{k-1}) = 0$.

So in particular $\mathbb{P}(E) \to B$ satisfies hypotheses of Leray–Hirsch; it follows that

$1, h_p, \ldots, h_p^{k-1}$ generate $H^*(\mathbb{P}(E);\mathbb{Z})$ as a $H^*(B;\mathbb{Z})$-module.

(module action: $\underset{\substack{\cap \\ H^*(B)}}{b} \cdot \underset{\substack{\cap \\ H^*(\mathbb{P}(E))}}{e} := \pi^*(b) \cup e$).

So every element $e \in H^*(\mathbb{P}(E))$ can be written as $\sum_{j=0}^{k-1} \pi^*(b_j) \cup h_p^j$ for unique $b_j \in H^*(B;\mathbb{Z})$.

$$H^{2k}(\mathbb{P}(E);\mathbb{Z}).$$

Consider the element $\overset{\substack{H^{2k}(\mathbb{P}(E);\mathbb{Z}) \\ \cup\!\!\!\text{I}}}{h_p^k}$. (note: if $E \to B$ trivial bundle, then $\begin{array}{c} E = \mathbb{C}^k \times B \\ \downarrow \\ B \end{array}$ so $\begin{array}{c} \mathbb{P}(E) = \mathbb{C}\mathbb{P}^{k-1} \times B \\ \downarrow \\ B \end{array}$, & $L \to \mathbb{P}(E)$ is $\pi^*_{\mathbb{C}\mathbb{P}^{k-1}}(L_{taut})$ where $\pi_{\mathbb{C}\mathbb{P}^{k-1}}: \mathbb{P}(E) \to \mathbb{C}\mathbb{P}^{k-1}$ exists when E is trivial. In that case $h_p = \pi^*_{\mathbb{C}\mathbb{P}^{k-1}} h$, and $h_p^k = \pi^*_{\mathbb{C}\mathbb{P}^{k-1}}(h^k) \equiv 0$).

Leray–Hirsch $\Rightarrow$ there exists a relation of the form

$$(\star)\quad h_p^k + \pi^*(a_1)\smile h_p^{k-1} + \cdots + \pi^*(a_k)\smile h_p^0 = 0.$$

for unique classes $a_1 \in H^2(B;\mathbb{Z})$, $a_2 \in H^4(B;\mathbb{Z})$, $\ldots$, $a_k \in H^{2k}(B;\mathbb{Z})$.

Def: $c_i(E) := a_i$ as given above, $\in H^{2i}(B;\mathbb{Z})$.. ith Chern class.

(By convention $c_0(E) = 1$, coeff. of h_p^k in relation above; & note $c_i(E) = 0$ for $i > \mathrm{rank}_{\mathbb{C}}(E)$).

Since $h_p^k \equiv 0$ when E is trivial $\Rightarrow$ each a_i hence $c_i(E) = 0$).

(real case: Have $h_p \in H^1((\mathbb{R})\mathbb{P}(E);\mathbb{Z}/2)$. Leray-Hirsch using $1, h_p, \ldots, h_p^{k-1}$ applies, so ($k = \mathrm{rank}_{\mathbb{R}}(E)$)
$\exists!$ classes $a_i \in H^i(B;\mathbb{Z}/2)$ so that $h_p^k + \pi^*(a_1)\smile h_p^{k-1} + \cdots + \pi^*(a_k)\smile h_p^0 = 0$.
$\Rightarrow$ define ith Stiefel-Whitney class $w_i(E) := a_i \in H^i(B;\mathbb{Z}/2)$.).

Properties: (real case parallel - exercise)

Naturality?

Say $f: A \to B$ & consider f^*E.

$$\begin{tikzcd} & f^*E \arrow[d] & & E \arrow[d] \\ & A & & B \end{tikzcd}$$

Note that $\mathbb{P}(f^*E) = f^*\mathbb{P}(E)$, and we have a map

$$\begin{tikzcd} f^*\mathbb{P}(E) \arrow[r, "f"] \arrow[d] & \mathbb{P}(E) \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}, \quad \& \quad \begin{tikzcd} L \arrow[d] \\ \mathbb{P}(f^*E) \end{tikzcd} = f^*\left(\begin{tikzcd} L \arrow[d] \\ \mathbb{P}(E) \end{tikzcd}\right).$$

So h_p in $H^2(\mathbb{P}(f^*(E)))$ is $f^* h_p$ (from $H^2(\mathbb{P}(E))$).

$\Rightarrow$ applying f^* to $(\star)$ gives in $H^*(\mathbb{P}(f^*E))$ the following relation:

$$\underset{\overset{\shortparallel}{f^*h_p}}{h_p^k} + \pi^*(f^*a_1)\smile h_p^{k-1} + \cdots + \pi^*(f^*(a_k))\smile h_p^0 = 0.$$

we conclude $c_i(f^*E) = f^*a_i = f^*c_i(E)$. ✓.

Does this recover the usual definition when $k=1$?

$L \to B$ line bundle (complex), i.e, L_b is 1-dim'l, and $\mathbb{P}(L_b)$ is a point.

i.e, $\pi: \mathbb{P}(L) \xrightarrow{\cong} B$ is a homeomorphism w/ fibers $\mathbb{CP}^0 = \text{point}$.

And moreover the tautological bundle $L_{taut} \to \mathbb{P}(L)$ corresponds under homeo. (meaning $\cong \pi^*$ of) to $L \to B$ we started with.

$$\Rightarrow h_p := -c_1^{old}(\underset{\overset{\uparrow}{H^2(\mathbb{P}(E);\mathbb{Z})}}{L_{taut}}) = -c_1^{old}(L) \in H^2(B;\mathbb{Z})$$

In $H^*(\mathbb{P}(E);\mathbb{Z})$, $h_p^0 = 1$ is a basis for $H^*(\mathbb{P}(E);\mathbb{Z})$ as a $H^*(B;\mathbb{Z})$ module.

so have a relationship

$$h_p^{\overset{\text{rank}(L)}{1}} + \overset{\text{an iso.}}{\pi^*}(c_1^{new}(L))\smile \overset{\overset{1}{\equiv}}{h_p^0} = 0 \quad \text{for some class } c_1^{new}(L) \in H^2(B;\mathbb{Z}).$$

$$\Rightarrow \pi^* c_1^{new}(L) = -h_p = c_1^{old}(L_{taut}) = \pi^* c_1^{old}(L)$$

$$\Rightarrow c_1^{new}(L) = c_1^{old}(L). \quad \checkmark$$

Whitney sum formula?

(real case parallel again)

Say have E_1, E_2 complex vector bundles over B of complex ranks k, ℓ respectively.

Form $\begin{matrix} E_1 \oplus E_2 \\ \downarrow \\ B \end{matrix}$, which has sub-bundles $E_1, E_2 \subseteq E_1 \oplus E_2$ whose fibers are complementary vector spaces,

inducing $\mathbb{P}(E_1), \mathbb{P}(E_2) \hookrightarrow \mathbb{P}(E_1 \oplus E_2)$ and $\mathbb{P}(E_1) \cap \mathbb{P}(E_2) = \emptyset$.

(if V_1, V_2 complementary vector subspaces of V then $\mathbb{P}(V_1) \cap \mathbb{P}(V_2)$ is empty in $\mathbb{P}(V)$)

Let $U_1 = \mathbb{P}(E_1 \oplus E_2) - \mathbb{P}(E_1)$ $\quad U_2 = \mathbb{P}(E_1 \oplus E_2) - \mathbb{P}(E_2)$

claim: open set retracting onto $\mathbb{P}(E_2)$ + open set retracting onto $\mathbb{P}(E_1)$.

(why? $\mathbb{CP}^k \setminus \mathbb{CP}^i$ retracts onto $\mathbb{CP}^{k-i-1}$

$[x_0 : \cdots : x_k]$; $[x_0 : \cdots x_i : 0 \cdots 0]$; $[0 : \cdots : 0 : x_{i+1} : \cdots x_k]$)

Also, L_{taut} on $\mathbb{P}(E_1 \oplus E_2)$ restricts to L_{taut} on each $\mathbb{P}(E_i)$.

$$\Rightarrow h_{\mathbb{P}(E_1 \oplus E_2)} \Big|_{\mathbb{P}(E_i)} = h_{\mathbb{P}(E_i)}.$$

Let $\omega_1 = \sum_{j=0}^{k} \pi^* c_j(E_1) \smile h_{\mathbb{P}(E_1 \oplus E_2)}^{k-j}$ $\qquad$ $\omega_2 = \sum_{j=0}^{\ell} \pi^* c_j(E_2) \smile h_{\mathbb{P}(E_1 \oplus E_2)}^{\ell - j}$

rank(E_1) ; using $c_0(E_i) = 1$; rank(E_2)

$$= h_{\mathbb{P}(E_1 \oplus E_2)}^{k} + \pi^* c_1(E_1) \smile h_{\mathbb{P}(E_1 \oplus E_2)}^{k-1} + \cdots \qquad = h_{\mathbb{P}(E_1 \oplus E_2)}^{\ell} + \pi^* c_1(E_2) \smile h_{\mathbb{P}(E_1 \oplus E_2)}^{\ell-1} + \cdots$$

By definition, $\omega_1|_{\mathbb{P}(E_1)} \equiv 0$ ($h_{\mathbb{P}(E_1 \oplus E_2)}$ restricts to $h_{\mathbb{P}(E_1)}$); similarly $\omega_2|_{\mathbb{P}(E_2)} \equiv 0$.

So, ω_1 induces a class $\tilde{\omega}_1 \in H^{2k}(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_1)) \cong H^{2k}(\mathbb{P}(E_1 \oplus E_2), U_2)$.

$\mathbb{P}(E_1) \simeq U_2 \subseteq \mathbb{P}(E_1 \oplus E_2) \setminus \mathbb{P}(E_2)$

Also, ω_2 induces a class $\tilde{\omega}_2 \in H^{2\ell}(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_2))$
$$\cong$$
$$H^{2\ell}(\mathbb{P}(E_1 \oplus E_2), U_1).$$

Using the relative version of the cup product, $\exists$ a comm. diagram

$U_1 \cup U_2 = \mathbb{P}(E_1 \oplus E_2)$
$(\mathbb{P}(E_1) \cap \mathbb{P}(E_2) = \emptyset)$.
so this group is 0!

$$\begin{tikzcd}
\omega_1 \quad , \quad \omega_2 \arrow[r, mapsto] \arrow[ddd, dashed, bend right] & {} \arrow[d] \\
H^{2k}(\mathbb{P}(E_1\oplus E_2),U_2)\times H^{2l}(\mathbb{P}(E_1\oplus E_2),U_1) \arrow[r, "\cup"] \arrow[d] & H^{2k+2l}(\mathbb{P}(E_1\oplus E_2),U_1\cup U_2) \arrow[d] \arrow[d, shift left=3, "\text{so } \tilde{\omega}_1\cup\tilde{\omega}_2=0"] \\
H^{2k}(\mathbb{P}(E_1\oplus E_2))\times H^{2l}(\mathbb{P}(E_1\oplus E_2)) \arrow[r, "\cup"] & H^{2k+2l}(\mathbb{P}(E_1\oplus E_2)) . \\
\omega_1 \quad , \quad \omega_2 \arrow[r, mapsto] & \omega_1\cup\omega_2 = ?
\end{tikzcd}$$

image of $\tilde{\omega}_1 \cup \tilde{\omega}_2$.

$= 0,$

So. $\omega_1 \cup \omega_2 = 0$; expanding this out we get a relation:

$h_p^{k+l} + \cdots\cdots = 0$ coming from cupping $\omega_1 \cup \omega_2$.

coeff. of h_p^{k+l-j} is $\sum \pi^*(c_i(E_1) \cup c_{j-i}(E_2))$.

$\Rightarrow \quad c_j(E_1\oplus E_2) = \sum_i c_i(E_1) \cup c_{j-i}(E_2)$ as desired. ☒ .