Math 51 Homework 4

Due Friday July 15, 2016 by 1 pm

Instructions: Complete the following problems. Late homework will not be accepted. Please be sure to review the expectations for your submitted homework outlined online (such as: always including your name and ID number on the homework, stapling your homework, and guidelines for write-ups which will receive full credit). *Make sure to submit your homework to the correct person; (if you are in Section 01, submit to Zev, and if in Section 02, submit to Valentin).*

Part I: Book problems: From Levandosky's *Linear Algebra*, do the following exercises:

- Section 16: # 4, 9, 12, 16
- Setion 17: # 3, 6, 15*
  *Note for # 15: in part b, compute the area two ways — with and without det A.
- Section 21: # 1,3, 8, 11, 12,
- Section 22: # 2
- Section 23: # 1, 2, 7, 8, 10.

Part II: Non-book problems:

1. Recall (see Levandosky, p. 90) that a *diagonal matrix D* is a square matrix whose entries *not* located on the main diagonal are all equal to zero; that is, the matrix takes the form

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

(a) The product of two diagonal matrices (of the same size) is especially easy to compute. Compute the following product:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

(b) Under precisely what conditions does a diagonal matrix have an inverse? Find a formula for the inverse of a diagonal matrix in terms of its diagonal entries $d_1, d_2, \ldots, d_n$.

2. The circle of radius 1 centered at $(x, y) = (1, 2)$ has equation

$$(x-1)^2 + (y-2)^2 = 1. \qquad (1)$$

(a) Let u, v be the linear coordinates on $\mathbb{R}^2$ with respect to the basis $\mathcal{B} = \{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]\}$. Express u and v in terms of the standard coordinates x and y, and also express x and y in terms of u and v. Use the latter to express the above equation (1) in terms of u and v. What is the graph of the resulting object in the uv-plane?

(b) For the basis of part (a), determine the matrix (called C^{-1} in Levandosky) used to transform $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ into $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ and identify in terms of one of the linear transformations listed in L14. Is it consistent with the object you described in part (a)?

(c) Redo part (a), but now with u, v as the coordinates on $\mathbb{R}^2$ with respect to the basis $\mathcal{B} = \{\left[\begin{smallmatrix} 3 \\ 7 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]\}$. (You don't have to draw the graph of the u, v equation you obtain from (1)).

3. Let $L = \text{span}(\left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right])$. Find an *orthonormal basis* $\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2\}$ for $\mathbb{R}^2$ with the property that $\mathbf{w}_1$ lies in L.

4. Let A be an $n \times n$ matrix, and T its associated linear transformation. For p a positive integer, let A^p denote the matrix obtained by multiplying A by itself p times (for example, $A^1 = A$, $A^2 = AA$, etc.). Similarly, we denote by T^p the p-fold composition $\underbrace{T \circ T \circ \cdots \circ T}_{k \text{ times}}$ (the notation T^p will not factor into this problem but it is helpful notation to introduce).

(a) Suppose $\mathbf{v}_1$ is an eigenvector of A with eigenvalue λ_1. Show that $\mathbf{v}_1$ is also an eigenvector of the matrix A^P and find its eigenvalue.

(b) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be eigenvectors of A corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Suppose that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

Find $A^p\mathbf{v}$ (as a linear combination of the eigenvectors $\mathbf{v}_i$).

5. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that has eigenvector $\mathbf{v}_1 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ with eigenvalue $\lambda_1 = -2$, and eigenvector $\mathbf{v}_2 = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ with eigenvalue $\lambda_2 = 1$.

(a) What is the matrix of T with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$?

(b) If $\mathbf{x} = \left[\begin{smallmatrix} 4 \\ -2 \end{smallmatrix}\right]$, find the coordinates $[x]_\mathcal{B}$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$.

(c) Find A, the matrix of T in standard coordinates.

(d) Find A^7. (*Hint*: use the fact that A is *diagonalizable*, meaning it is *similar* to a diagonal matrix.)

6. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Does A have an eigenbasis; i.e., does there exist a basis for $\mathbb{R}^3$ consisting entirely of eigenvectors of A? Explain.

7. Let A be a 3×3 matrix with eigenvalues $-1, 3, 0$.

(a) Is A invertible? Explain your answer.

(b) Does A have an eigenbasis? Explain your answer.