Math 51 Homework 5

Due Friday July 22, 2016 by 1 pm

Instructions: Complete the following problems. Late homework will not be accepted. Please be sure to review the expectations for your submitted homework outlined online (such as: always including your name and ID number on the homework, stapling your homework, and guidelines for write-ups which will receive full credit). *Make sure to submit your homework to the correct person; (if you are in Section 01, submit to Zev, and if in Section 02, submit to Valentin).*

Part I: Book problems: From **L**evandosky's *Linear Algebra* and **C**olley's *Vector Calculus*, do the following exercises:

- Section **L**18: # 4
- Section **L**25: #5, 6, 12, 15
- Section **C**2.1: # 12, 16, 20, 38 (for 20: in the language developed in class, first draw a *contour map*, which is a collection of level sets or *level curves* according to Colley and then sketch the graph)
- Section **C**2.2: # 8, 16, 50 (note: problem 50 will require the "Rigorous definition of Limit" given in in Definition 2.2 on page 100).
- Section **C**2.3: # 2, 4, 16, 20, 28, 30, 38, 58, 59 (note that an answer for 59 is given in the back, but we are asking you to prove the assertion there, and show all of your work)
- Section **C**2.4: # 22

Part II: Non-book problems:

1. Find the tangent line to the image of the given parametrized curve at the indicated point.
(a) $\mathbf{g}(t) = (e^t, \cos t^2, \ln(t+1))$ at $(1,1,0)$.
(b) $\mathbf{g}(t) = (t^2 - 5t + 6, t^2 - 4, t^2 - 9)$ at $(0,0,-5)$.

2. Find parametrizations for the following curves in $\mathbb{R}^2$:
(a) the ellipse centered at $(0,0)$ with principal axes oriented along the x- and $y-$axes, of lengths a and $2b$ respectively. (*hint*: the equation of the ellipse is $(\frac{x}{a})^2 + (\frac{y}{2b})^2 = 1$.)
(b) the circle with radius r and center at (c,d).

3. Consider the parametrized curve $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2$ defined by $\mathbf{g}(t) = \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right)$.
(a) Show that every point in the image of $\mathbf{g}$ lies on the hyperbola $x^2 - y^2 = 1$.
(b) Are there points on the hyperbola $x^2 - y^2 = 1$ that are not in the image of $\mathbf{g}$? Explain.

4. Let $f(x,y) = 1 - x^2 - y^2$. In Colley §2.3,[1] the partial derivatives f_x and f_y of any such function were shown to determine two lines tangent to the graph of f

$$\Gamma_f = \{(x,y,z) | z = f(x,y) = 1 - x^2 - y^2\}$$

[1] See p. 118-119, particularly the discussion above Theorem 3.3.

at a given point $(a, b, f(a, b))$. Namely, it was argued that the vectors $\begin{bmatrix} 1 \\ 0 \\ f_x(a,b) \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ f_y(a,b) \end{bmatrix}$ both give vectors which are parallel to tangent lines to Γ_f at $(a, b, f(a, b))$.

(a) Using the above discussion, find the parametric representation of two different lines L_1 and L_2 which are tangent to the surface $z = 1 - x^2 - y^2$ at the point $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

(b) Assuming it exists, the *tangent plane* to this surface at $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (also referred to as "the tangent plane to the graph of f at $(x, y) = (-\frac{1}{2}, \frac{1}{2})$" contains both lines L_1 and L_2, so it is uniquely determined. Specify this tangent plane in parametric form.

(c) Find an equation for this plane, expressed in the form $z = ax + by + c$. (Note: you can use either your answer to part (a) *or* the formula for tangent plane as found on page 119).

5. *Change of basis with orthogonal matrices.* An *orthogonal matrix* is an $n \times n$ matrix C whose columns form an *orthonormal basis* of $\mathbb{R}^n$. If C is an orthogonal matrix, than an important fact is that computing the inverse of C is very easy:

$$C^{-1} = C^T.$$

(you can also read about this in Levandosky p. 164 - 165). This is because, if

$$C = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \ddots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}$$

with $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ an orthonormal basis, then

$$C^T = \begin{bmatrix} - & \mathbf{v}_1^T & - \\ - & \mathbf{v}_2^T & - \\ \vdots & \ddots & \vdots \\ - & \mathbf{v}_n^T & - \end{bmatrix}$$

and hence

$$(C^T C)_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} ||\mathbf{v}_i||^2 = 1 & i = j \\ 0 & i \neq j \end{cases}$$

meaning, $C^T C = I_n$, so $C^T = C^{-1}$. This makes change of basis formulaes involving orthonormal bases particularly easy, because if $\mathcal{B}$ is an orthonormal basis of $\mathbb{R}^n$, then the change of basis matrix $C_{\mathcal{B}}$ is an orthogonal matrix, so $C_{\mathcal{B}}^{-1} = C_{\mathcal{B}}^T$.

Now, let $L = \text{span}(\left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right])$. It is a fact that you may have determined in HW last week that $\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2\} = \{\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}\}$ is an orthonormal basis for $\mathbb{R}^2$ with the property that $\mathbf{w}_1$ lies in L.

(a) Find the matrix of the linear transformation $Refl_L$ in $\mathcal{B}$-coordinates.

(b) Using the above fact, and the change of basis formula, find the matrix of the linear transformation $Refl_L$ in standard coordinates.

6. *Level sets of a particular quadratic form.* Let A be an $n \times n$ symmetric matrix. The *quadratic form* associated to A is a particular multivariable real-valued function defined as follows:

$$Q_A : \mathbb{R}^n \to \mathbb{R}$$

defined by the rule $Q_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = (A\mathbf{x}) \cdot \mathbf{x}$. Our goal is to sketch, without the aid of a calculator or computer, the level set $Q_A^{-1}(c)$ for a particular matrix A at a particular height c.

(a) Let $A = \begin{bmatrix} 17 & -6 \\ -6 & 8 \end{bmatrix}$. Write out $Q_A(\mathbf{x})$ as a function of x and y.

(b) It is a fact (which you needn't prove) that the matrix A in part (a) has eigenvectors $\left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$. Use this to find the eigenvalues of A, and to find an *orthonormal* eigenbasis $\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2\}$.

(c) Let u, v be $\mathcal{B}$-coordinates; so the expression $(u, v)_{\mathcal{B}}$ means $u\mathbf{w}_1 + v\mathbf{w}_2$. Use the information you found in (b) to write a simple expression for Q_A in terms of the $\mathcal{B}$ coordinates (use u, v) (*Hint*: it's not necessary to work through any messy algebra to do this.)

(d) Using the u, v expression you found in (c), sketch the graph of the ellipse with equation "$Q_A = 100$" (which is the *level set* $Q_A^{-1}(100)$) in a u, v coordinate systemm; be sure to label the u and v coordinate axes and their scales. (*Hint*: the ellipse should have major and minor axes parallel to your coordinate axes)

(e) Now in a new sketch, draw and label a "standard" x, y coordinate system. Then, in this system, draw and label both the "u-axis" and the "v-axis," indicating their positive and negative directions. (*Hint*: the u-axis satsifies $v = 0$, which you could conver to an expression of x and y; or alternatively the u-axis is simply span($\mathbf{w}_1$). The v-axis is handled analogously).

(f) Use the results of part (d) to sketch the level set $Q_A^{-1}(100)$, which is the ellipse with equation $Q_A = 100$ in standard coordinates, on the same x, y, coordinate system you used for your answer in part (e).