Math 51 Homework 7

Due Thursday, August 4, 2016 by 1 pm

Instructions: Complete the following problems. Late homework will not be accepted. Please be sure to review the expectations for your submitted homework outlined online (such as: always including your name and ID number on the homework, stapling your homework, and guidelines for write-ups which will receive full credit). *Make sure to submit your homework to the correct person; (if you are in Section 01, submit to Zev, and if in Section 02, submit to Valentin).*

NOTE: In all exercises (and exam problems) involving the Second derivative test: you could of course use Colley's Theorem 2.3 (p. 267) (the same as our "Second Derivative Test" theorem in class), which tells you that at a critical point $\mathbf{a}$, if $Q_{Hf(\mathbf{a})}$ is positive definite, then f has a local minimum and so on. However, please *do not use* and in fact *ignore* the "principal minors method" in Colley's blue box titled "Second derivative test for local extrema" (the blue box on p. 268). Instead, you should use Levandosky Chapter 26 (Prop. 26.1): you can figure out the definiteness of $Q_{Hf(\mathbf{a})}$ by looking at the signs of the eigenvalues of $Hf(\mathbf{a})$. *NOTE there is one exception to the above rule: you can use principal minors methods for* **C***4.2 problem #22.*

Also: in any exercise (and on any exam), please *do not* use Levandosky Prop. 26.2, which gives an ad hoc quirk method to determine the definiteness of a quadratic form in the 2-dimensional case. The point is to carry out a method that works in any dimension, using Levandosky Prop. 26.1 (which classifies quadratic forms according to the signs of eigenvalues).

Part I: Book problems: From **L**evandosky's *Linear Algebra* and **C**olley's *Vector Calculus*, do the following exercises:

- Section **L**26: #2, 6, 18
- Section **C**4.1: #10, 14, 18
- Section **C**4.2: # 4, 22, 36, 38, 40, 48 (in 36 and 38, to find boundary maxima and minima, you may want to *parametrize the boundary* via one or more parametric curve(s) $\mathbf{r} : I \to \mathbb{R}^2$, and consider the maxima and minima of $f \circ \mathbf{r}$, which gives the values of f on the image of $\mathbf{r}$; as in our examples given in class)
- Section **C**4, True/False Exercises (p. 306 of Colley): # 18 (please justify your work)

Part II: Quadratic form problems:

For each of the following symmetric matrices A,

(a) compute the associated quadratic form $Q_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ in standard coordinates $x_1, \ldots, x_n$,
(b) Find the (quadratic or cubic, depending on n) characteristic polynomial $p_A(\lambda)$ and the eigenvalues of A,
(c) Find an *orthonormal eigenbasis* $\mathcal{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ for A;
(d) compute the (simplified) expression for Q_A in terms of $\mathcal{B}$-coordinates $u_1, \ldots, u_n$; and
(e) From (d), determine the definiteness of Q_A.

1. $A = \begin{bmatrix} 5 & 6 \\ 6 & 10 \end{bmatrix}$ (so $n = 2$),

2. $A = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (so $n = 3$)

Part III: Domains of $\mathbb{R}^n$:

For each of the following domains $\mathcal{D} \subset \mathbb{R}^n$, determine:

(a) whether $\mathcal{D}$ is *closed* or *not closed*,

(b) whether $\mathcal{D}$ is *bounded* or *not bounded*,

(c) whether every continous function $f : \mathcal{D} \to \mathbb{R}$ attains a (global) maximum and minimum; meaning either (i) cite a Theorem guaranteeing that every continuous function attains a maximum and minimum or (ii) produce a continuous function $f : \mathcal{D} \to \mathbb{R}$ which does not attain its maximum or does not attain a minimum.

1. $\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3 | 1 \leq (x-1)^2 + (y+1)^2 + z^2 \leq 4\}$

2. $\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3 | 1 \leq x^2 + y^2 \leq 4\}$

3. $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 | -1 < x < 3, 3 \leq y \leq 5\}$

Helpful references for Part III: Colley Section 2.2. Definition 2.3 (p. 102) for the definition of a *closed set*, and the definition of a ball (p. 101), Colley p. 271 below Definition 2.4 for the definition of *bounded* (not in boldface), and Definition 2.4 for the definition of *compact.* Also, the *extreme value theorem*, Theorem 2.5 (p. 271)

Part IV: Other non-book problems:

1. Let A be a symmetric $n \times n$ matrix, and suppose that $Q = Q_A : \mathbb{R}^n \to \mathbb{R}$ is the quadratic form given by

$$\begin{aligned} Q(\mathbf{x}) &= \mathbf{x} \cdot (A\mathbf{x}) \\ &= \mathbf{x}^T A \mathbf{x}. \end{aligned}$$

Show that for every $\mathbf{b} \in \mathbb{R}^n$,

(a) $\nabla Q(\mathbf{b}) = 2A\mathbf{b}$, and

(b) $HQ(\mathbf{b}) = 2A$.

(c) Using the above, and problem **C**2.3 # 59 (solved on HW5), verify the following fact: if $f : \mathbb{R}^n \to \mathbb{R}$ is a (sufficiently differentiable) function, then its second-order Taylor approximation at $\mathbf{a} \in \mathbb{R}^n$

$$\begin{aligned} T_2(\mathbf{x}) &= f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a}) \\ &= f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2} Q_{Hf(\mathbf{a})}(\mathbf{x} - \mathbf{a}) \end{aligned}$$

satisfies:

$$\begin{aligned} T_2(\mathbf{a}) &= f(\mathbf{a}) \\ DT_2(\mathbf{a}) &= Df(\mathbf{a}) \\ HT_2(\mathbf{a}) &= Hf(\mathbf{a}) \end{aligned}$$