Math 51 Homework 8

Due Wednesday August 10, 2016 by 1 pm

Instructions: Complete the following problems. Late homework will not be accepted. Please be sure to review the expectations for your submitted homework outlined online (such as: always including your name and ID number on the homework, stapling your homework, and guidelines for write-ups which will receive full credit). *Make sure to submit your homework to the correct person; (if you are in Section 01, submit to Zev, and if in Section 02, submit to Valentin).*

Part I: Book problems: From **L**evandosky's *Linear Algebra* and **C**olley's *Vector Calculus*, do the following exercises:

- Section **C**4.3, #6, 8, 12, 22, 26, 30, 32 (note: problems 12, 30, and 32 may involve *Lagrange multipliers with multiple constraints.* Or, another option may be, when the intersection of the contraints form a curve, to parametrize the curve.)
- Section **C**4.4, #14,
- Section **C**4, Miscellaneous Exercises (p. 306 of Colley): #14, 24.

Part II: Distance to a closed set

1. In this problem, we'll show that every nonempty closed subset of $\mathbb{R}^n$ (not necessarily bounded!) has a "closest distance to the origin." That is, suppose S is a closed, non-empty subset of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(\mathbf{x}) = ||\mathbf{x}||$; we'll show that the restriction of f to S *always* attains a (global) minimum value.
First, since S is non-empty, let's pick some point $\mathbf{a}$ lying in S, and let $R = ||\mathbf{a}|| = f(\mathbf{a})$; now let's define the following sets containing $\mathbf{a}$:

$$S_{near} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} \in S \text{ and } ||\mathbf{x}|| \leq R\} = \{\mathbf{x} \in S : ||\mathbf{x}|| \leq R\}$$
$$S_{far} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} \in S \text{ and } ||\mathbf{x}|| \geq R\} = \{\mathbf{x} \in S : ||\mathbf{x}|| \geq R\}.$$

(a) Show that S_{near} is closed and bounded. (*Hint*: the intersection of any two closed sets is closed, so to show S_{near} is closed it suffices to describe S_{near} as an intersection of two closed sets). Thus, by the Extreme Value Theorem, the restriction of f to S_{near} attains a minimum value.
(b) Show that the restriction of f to S_{far} attains a minimum value.
(c) Use (a) and (b) to explain why you know that the restriction of f to S attains a minimum value.

Remark: An argument identical to this one establishes that there is a "minimal distance" from a closed set S to any point $p \in \mathbb{R}^n$, not just to the origin $p = \mathbf{0}$.

2. Let P be the plane given by the equation $ax + by + cz = d$, and q be the point (x_0, y_0, z_0). We call the minimal distance from any point on P to q the *distance from P to q*; you may take for granted this quantity exists (by essentially the work you did in the previous problem!)

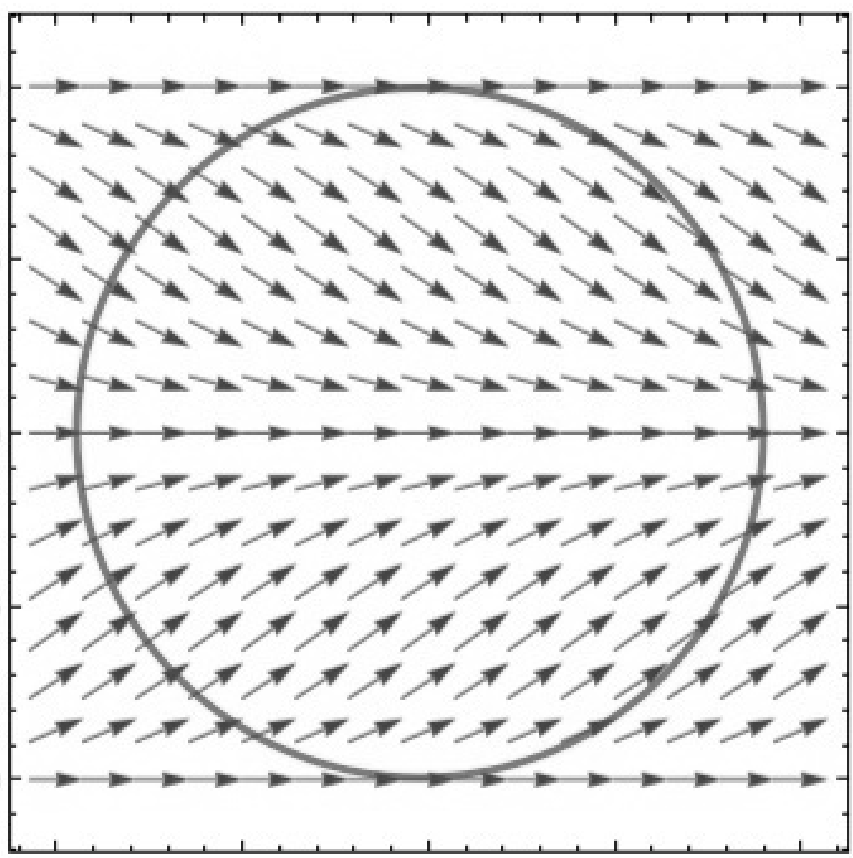
Show that the distance from P to q is given by the formula

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Hint: as we will do in class, it is often easier (in terms of taking derivatives) to minimize the *squared distance function*; then note that square distance is minimized precisely when the usual distance function is minimized.

Part III: Other non-book problems:

1. The gradient of a function $f(x, y)$ is indicated by arrows at various points in the xy-plane, drawn below. Identify the approximate location of the points on the circle where the restriction of f to the circle attains its maximum and minimum. Mark these points on the picture below; explain your answer. (Then, don't forget to turn in this sheet with your other problems!)



2. Consider the ellipse E in the xy-plane having equation $(\frac{x}{3})^2 + (\frac{y}{4})^2 = 1$. A rectangle is said to be *inscribed* in E if all four vertices live on the curve E; it follows that the four sides of such a rectangle must be parallel to the coordinate axes (you don't have to prove this). Determine the inscribed rectangle having maximum area. (Make sure to explain how you know that your extremum is the "global maximum" you seek).

3. Let B be the ball defined by $B = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 24\}$ and suppose $f(x, y, z) = 2x + y - z$.

(a) Explain why the restriction of f to B must atain both a maximum and a minimum value.

(b) Compute the points in B where these maximum and minimum values are attained.