Math 51 Final Exam — August 13, 2016

Name: ______________________________ **SUID#:** ____________

Circle your section:	
Section 01 (1:30-2:50PM)	Section 02 (3:00-4:20PM)

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text (unless otherwise stated on the syllabus and in class). If doing so, be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 17 numbered pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 3 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- Paper not provided by teaching staff is prohibited. If you need extra room for your answers, use the back side of the question page or other extra space provided at the front of this packet, and clearly indicate that your answer continues there. Do not unstaple or detach pages from this exam.
- Please show your work unless otherwise indicated. *Do not make multiple guesses: if there are multiple answers given to a question that only asks for one answer, your grade will be the minimum of the possible scores you would have received.*
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the Stanford's honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	**Total**
Points:	12	15	16	14	10	16	12	12	12	14	16	12	14	175
Score:														

1. (12 points) You are a house building construction company. The size of a house you can build (in terms of square feet of floorspace) is described by the following production function

$$h(x, y) = 100x^{1/2}y^{1/4}$$

where x denotes units of wood used and y denotes units of nails used in construction (x and y need not be whole numbers). Suppose that wood costs \$5 per unit, and nails cost \$10 per unit. Your company is building a new house, and has a total maximum budget for wood and nails of \$1000. What is the largest size house your company can build?

You may assume that there *is* a largest size house that you can build given your company's budget, and that it occurs precisely when you spend your entire budget. You may express your answer in the form $100x_0^{1/2}y_0^{1/4}$ for some x_0, y_0.

2. (15 points) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $g(x, y) = \frac{3}{2}x^2 + \frac{1}{2}y^2 - x^3y - 2$. Find all critical points of g and classify each one of them as a local minimum, local maximum, or saddle point.

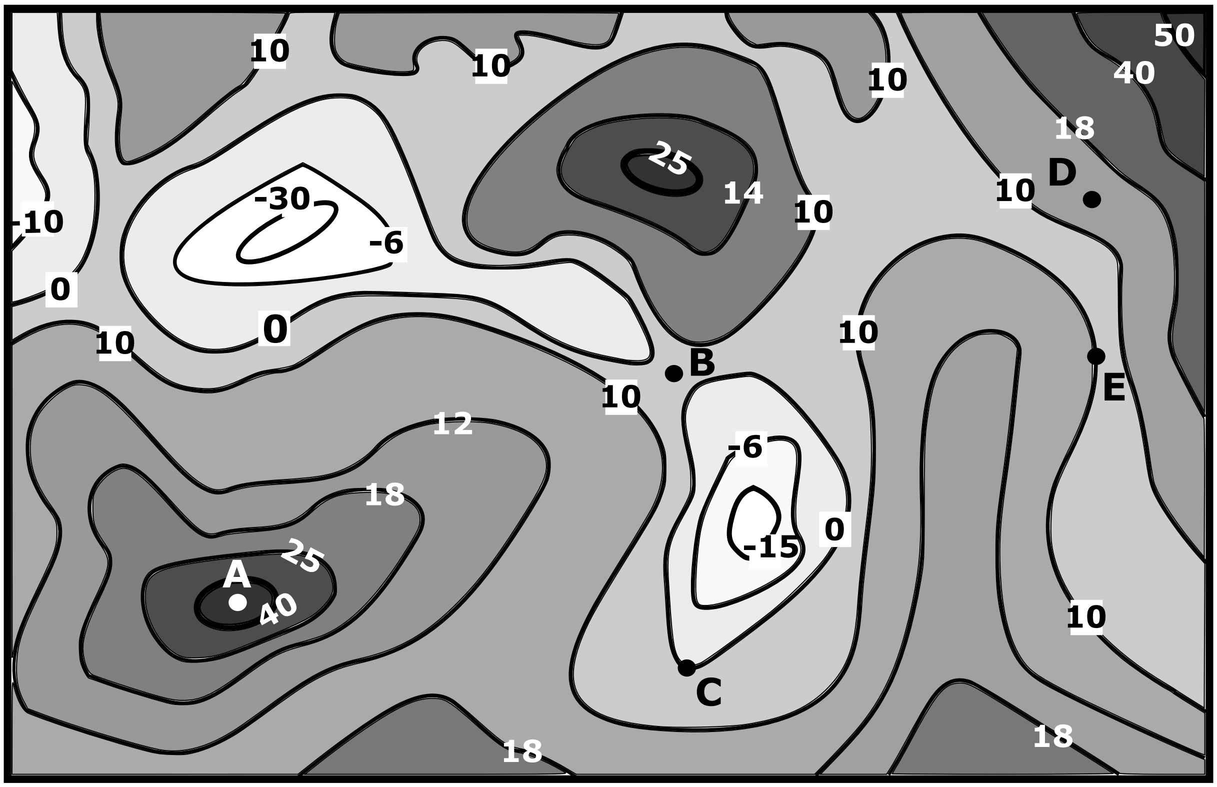
3. (a) (3 points) Say what it means for a set $S \subset \mathbb{R}^n$ to be a *subspace of* $\mathbb{R}^n$.

(b) (5 points) Let $C = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 | 2x + 3y - z = 0 \text{ and } x \geq 0\} \subset \mathbb{R}^3$. Is C a subspace of $\mathbb{R}^3$? Justify your answer completely.

(c) (3 points) Give the definition of the *dimension* of a subspace $S \subset \mathbb{R}^n$.

(d) (5 points) Calculate, with justification, the the dimension of the subspace $X = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \\ -8 \end{bmatrix}\right)$ of $\mathbb{R}^4$? (you may take for granted that X, being the span of a collection of vectors, *is* in fact a subspace).

4. *This question does not require any justification for your answers.* Below is a contour map of a function $f : \mathbb{R}^2 \to \mathbb{R}$, e.g., a collection of level curves. You may assume that f has continuous first and second derivatives, and that the scales on the x and y axes, which are parallel to the edges of the box, are the same. The numbers indicated are the heights of the various level sets, and the contour map is also (roughly) shaded with darker regions representing points where f is relatively higher, and lighter regions representating points where f is relatively lower.



(a) (2 points) Sketch, on the plot, the direction of steepest decrease of f at the point C.

(b) (2 points) Sketch, on the plot, the approximate location of a critical point which is a saddle point which is not any of A, B, C, D, or E. Label your point "F".

(c) (2 points) Sketch, on the plot, the approximate location of a critical point of f at which the Hessian of f is positive definite, which is not any of A, B, C, D, E, or F. Label your point "G".

(d) (2 points) (Circle one) At the point C the value of $\frac{\partial^2 f}{\partial x^2}(C)$ is POSITIVE ZERO NEGATIVE.

(e) (2 points) (Circle one) If $\mathbf{v}$ is the unit vector in the direction of $\left[\begin{smallmatrix} 2 \\ -2 \end{smallmatrix}\right]$, then at the point A, the directional derivative $D_{\mathbf{v}} f(A)$ is POSITIVE ZERO NEGATIVE.

(f) (2 points) (Circle one) If $\mathbf{v}$ is the unit vector in the direction of $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, then at the point D, the directional derivative of the partial derivative $D_{\mathbf{v}}(\frac{\partial f}{\partial x})(D)$ is POSITIVE ZERO NEGATIVE.

(g) (2 points) (Circle one) At the point E the value of $\frac{\partial f}{\partial y}(E)$ is POSITIVE ZERO NEGATIVE

5. (10 points) You would like to understand the approximate relationship between the *number of floors a given building has*, x and the building's *height* y in meters. You examine the 3 buildings closest to you and find the number of floors x_i and the height y_i of each building to be:

- Building 1: $(x_1, y_1) = (1, 5)$
- Building 2: $(x_2, y_2) = (3, 10)$,
- Building 3: $(x_3, y_3) = (6, 20)$.

To try to understand the rough relationship between x and y, you would like to find a *linear regression* for the data you have collected , meaning a line $y = mx + b$ which *best fits* the data points you have collected, in the following sense: to a given line $y = mx + b$ which is determined by a slope m and a y-intercept b, you can associate an *error function* which measures the *deviation* from this line and the data points above, meaning its failure to match the points. In this case, the function is given in terms of m and b as follows:

$$E(m, b) = \sum_{i=1}^{3} (y_i - (mx_i + b))^2 = (5 - (m \cdot 1 + b))^2 + (10 - (m \cdot 3 + b))^2 + (20 - (m \cdot 6 + b))^2.$$

You may take for granted that there is a unique (m_0, b_0) that minimizes the function $E(m, b)$; the associated line $y = m_0 x + b_0$ is called the *best fit line* to the data points above.

Find, using calculus, the best fit line to the data points above. *Note:* there is no need to simplify m_0 and b_0 once you have found them. **Hint**: Since you are told a global minimum exists, you know the global minimum must be a local minimum and in particular a critical point of E. If E has only one critical point, there is therefore no need to perform a second derivative test.

(More space for your solution to problem 5, if necessary)

6. Consider the following quadratic form

$$q(x, y, z) = 2x^2 - 3y^2 - z^2 - 4yz.$$

(a) (4 points) Write down a symmetric matrix A whose associated quadratic form equals q; that is, a matrix with $A^T = A$ so that $q(x, y, z) = Q_A(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(b) (6 points) Determine with justification the definiteness of the quadratic form $q(x, y, z)$.

(c) (6 points) Set up, but do not solve, system of equations in x, y, z and maybe other unknown variables, which would find the local extrema (along with local saddle points) of $q(x, y, z)$ restricted to the region $S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x^2 - 2y^2 = 1, y^2 + z^2 = 4\}$. For full credit, you should write your system as a list of equations with as many equations as there are unknowns (so if x, y, and z are the only unknowns, you should write three equations).

7. (a) (8 points) Compute the second order Taylor approximation to the function

$$f(x, y) = \ln(x^2 + y^2)$$

at $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Express your answer as a polynomial in the form $a + b(x - \frac{1}{\sqrt{2}}) + c(y - \frac{1}{\sqrt{2}}) + d(x - \frac{1}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}) + e(x - \frac{1}{\sqrt{2}})^2 + f(y - \frac{1}{\sqrt{2}})^2$.

(b) (4 points) Use the result from part (a) to estimate the value of $\ln(1.02) = f(\frac{1}{\sqrt{2}} + 0.1, \frac{1}{\sqrt{2}} - 0.1)$.

8. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 5 & -3 & 2 \\ 2 & a & 1 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, for some a.

(a) (6 points) Under what conditions if any on a is $\mathbf{b} \in C(A)$? Justify your work.

(b) (6 points) Under what conditions on a are there

- no solutions,
- exactly one solution,
- many solutions

to $A\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$? (for each part your answer may be of the form $a = c$, $a \neq c$, $a \geq c$ or *never*). Justify your work.

9. (12 points) (1 point for each sub-part) *This question does not require you to show your work.* In each of the following parts, you will be given a function f and a region $E \in \mathbb{R}^n$. Say first (i) whether E is closed or not closed, (ii) whether E is bounded or not bounded, and (iii) whether f attains an absolute minimum on the region E.

   (a) $E = \{(x, y, z) \mid -1 \le x \le 2, 1 \le y \le 3, 5 \le z \le 7\}$. $f(x, y, z) = xyz - x^2ye^{xy}$.

      i. (circle one) E is CLOSED NOT CLOSED.

      ii. (circle one) E is BOUNDED NOT BOUNDED.

      iii. (circle one) The restriction of f to E ATTAINS DOES NOT ATTAIN an absolute minimum on the region E.

   (b) $E = \{(x, y, z, w) \mid x < 1, y < 2, (w + z) < 2, \} \subset \mathbb{R}^4$, $f(x, y, z, w) = 6 - w - z$.

      i. (circle one) E is CLOSED NOT CLOSED.

      ii. (circle one) E is BOUNDED NOT BOUNDED.

      iii. (circle one) The restriction of f to E ATTAINS DOES NOT ATTAIN an absolute minimum on the region E.

   (c) $E = \{(x, y, z) \mid 2x - y + z = 50\}$, $f(x, y, z) = x^2 + y^2 + z^2$.

      i. (circle one) E is CLOSED NOT CLOSED.

      ii. (circle one) E is BOUNDED NOT BOUNDED.

      iii. (circle one) The restriction of f to E ATTAINS DOES NOT ATTAIN an absolute minimum on the region E.

   (d) $E = \{(x, y) \mid x^2 - 4y^2 \le 5,\ |y| \le 10\}$, $f(x, y) = 2x + 3y$.

      i. (circle one) E is CLOSED NOT CLOSED.

      ii. (circle one) E is BOUNDED NOT BOUNDED.

      iii. (circle one) The restriction of f to E ATTAINS DOES NOT ATTAIN an absolute minimum on the region E.

10. (14 points) Let $f(x, y) = 2x^2 - x + 3y^2 + 4$. Find the absolute minimum and maximum value attained by f when restricted to the *disc*

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

You may assume that a global minimum and maximum of $f|_D$ exist.

11. Let $S = g^{-1}(5) \subset \mathbb{R}^3$ be the level set of a differentiable function g. Suppose you are told that the point $\mathbf{a} = (1, -1, 2)$ lies on surface S and morevoer that at $(1, -1, 2)$, there is a tangent plane to S which is given parametrically as

$$P = \{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \}.$$

(a) (5 points) Find an equation for the plane P in the form $ax + by + cz = d$.

(b) (6 points) Now, let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function given by $f(x, y, z) = \frac{3}{4}x^2 - xe^{y+1} + bzy$ for some scalar $b \in \mathbb{R}$. Show that it is not possible for the restriction $f|_S$ of f to S to have a local extremum at $(1, -1, 2)$ unless $b = \frac{1}{2}$. *Hint*: think about the what is required for $f|_S$ to have a local extremum at $(1, -1, 2)$ geometrically.

(c) (5 points) Now, let $\mathbf{r} : [0,1] \to \mathbb{R}^3$ be a differentiable parametric curve. Suppose that $\mathbf{r}(\frac{1}{2}) = (1,-1,2)$, and that $\mathbf{r}'(\frac{1}{2})(= D\mathbf{r}(\frac{1}{2})) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Under what conditions, if any, on b, does the composition $f \circ \mathbf{r}$ have a critical point at $\frac{1}{2}$?

12. Let A be a 2×2 matrix. Suppose you are not told A, but instead are given that

$$\text{rref}(2I - A) = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}, \quad \text{rref}(5I - A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{rref}(-I - A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

(a) (6 points) Calculate, with justification, a basis of $\mathbb{R}^2$ consisting of eigenvectors of A, and say what their corresponding eigenvalues are.

(b) (6 points) Determine, using any method of your choice, the matrix of A.

13. (14 points) (2 points each) Each of the statements below is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). For each part, decide which and circle the appropriate choice; you *do not* need to justify your answers.

(a) The linear transformation $Proj_L : \mathbb{R}^2 \to \mathbb{R}^2$ given by projecting to a line L is invertible. T F MAYBE

(b) If a 3×3 matrix A has characteristic polynomial $p(\lambda) = \lambda(\lambda - 1)(\lambda + 3)$, then A is diagonalizable. T F MAYBE

(c) Let A be an $n \times n$ matrix whose first two rows are equal. Then A is invertible. T F MAYBE

(d) If A is a 2×2 matrix, then $\det(A) = \det(\text{rref}(A))$. T F MAYBE

(e) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function with continuous first and second partials. If $\mathbf{a} \in \mathbb{R}^3$ is a critical point of f, and the Hessian matrix at $\mathbf{a}$, $Hf(\mathbf{a})$, has eigenvalues 3, 2, and 0, then $\mathbf{a}$ is a local maximum of f. T F MAYBE

(f) If A is a 4×2 matrix and B is a 2×4 matrix, then BA is the identity matrix. T F MAYBE

(g) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function whose partial derivatives $\frac{\partial f}{\partial x_i}$ are all defined at $\mathbf{a} \in \mathbb{R}^2$. Then f is differentiable at $\mathbf{a}$. T F MAYBE