Solutions to Math 51 Final Exam — August 13, 2016

1. (12 points) You are a house building construction company. The size of a house you can build (in terms of square feet of floorspace) is described by the following production function

$$h(x,y) = 100x^{1/2}y^{1/4}$$

where x denotes units of wood used and y denotes units of nails used in construction (x and y need not be whole numbers). Suppose that wood costs \$5 per unit, and nails cost \$10 per unit. Your company is building a new house, and has a total maximum budget for wood and nails of \$1000. What is the largest size house your company can build?

You may assume that there *is* a largest size house that you can build given your company's budget, and that it occurs precisely when you spend your entire budget. You may express your answer in the form $100x_0^{1/2}y_0^{1/4}$ for some x_0, y_0.

The question is asking us to find a global extremum of the function $h(x,y)$ given the constraint equation determined by our budget: the amount we spend should be exactly \$1000, meaning

$$5x + 10y = 1000;$$

in other words we are being asked to find the maximum of $h|_S$ where $S = g^{-1}(1000)$, where $g(x,y)$ is the function $g(x,y) = 5x + 10y$.

First, we calculate ∇h and ∇g:

$$\nabla h(x,y) = \begin{bmatrix} 50x^{-1/2}y^{1/4} \\ 25x^{1/2}y^{-3/4} \end{bmatrix}$$

$$\nabla g(x,y) = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Since $\nabla g(x,y) \neq \mathbf{0}$ (and in particular $\neq \mathbf{0}$ on S), we can apply the Lagrange Multipliers theorem to conclude that at a local extremum (x,y) of $f|_S$, for some $\lambda \in \mathbb{R}$

$$\nabla h(x,y) = \lambda \nabla g(x,y). \tag{1}$$

(To obtain full credit, one should check that $\nabla g(x,y) \neq 0$ on S so that Lagrange Multiplers are indeed applicable. Not checking this resulted in a minor deduction of 2 points). To find the local extrema (x,y), we solve the system in x, y and λ given by (1) and the constraint equation, namely:

$$50x^{-1/2}y^{1/4} = \lambda 5 \tag{2}$$

$$25x^{1/2}y^{-3/4} = \lambda 10 \tag{3}$$

$$5x + 10y = 1000 \tag{4}$$

We note that after multiplying the (2) by 2, the RHS expressions of (2) and (3) of become equal; which allows us to elminate λ and determine that 2 times the LHS of (2) equals the LHS of (3); that is

$$100x^{-1/2}y^{1/4} = 25x^{1/2}y^{-3/4};$$

multiplying both sides by $\frac{1}{25}x^{1/2}y^{3/4}$ we see that

$$4y = x. \tag{5}$$

We substitute (5) into the constraint equation (4) to find that $y = \frac{100}{3}$, which by (5) implies that, $x = \frac{400}{3}$.

Thus, we find only one potential local extremum of $h|_S$ at $(x,y) = (400/3, 100/3)$. Since we are told there is a global maximum, we conclude this point must induce the global maximum of $h|_S$. Hence, the largest size house our company can build is $h(400/3, 100/3) = \boxed{100(\frac{100}{3})^{1/2}(\frac{400}{3})^{1/4}}$.

2. (15 points) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $g(x, y) = \frac{3}{2}x^2 + \frac{1}{2}y^2 - x^3y - 2$. Find all critical points of g and classify each one of them as a local minimum, local maximum, or saddle point.

We have

$$\nabla g(x, y) = \begin{bmatrix} 3x - 3x^2y \\ y - x^3 \end{bmatrix}$$

To find the critical points, we set this equal to $\mathbf{0}$. This yields

$$3x(1 - xy) = 0$$

$$y = x^3$$

The first equation implies that either $x = 0$ or $1 = xy$. If $x = 0$, then the second equation yields $y = 0$, so we get the point $(0, 0)$. If $xy = 1$, then the second equation yields $x^4 = 1 \implies x = \pm 1$, so we get the points $(1, 1)$ and $(-1, -1)$. Now

$$Hg(x, y) = \begin{bmatrix} 3 - 6xy & -3x^2 \\ -3x^2 & 1 \end{bmatrix}$$

We compute

$$Hg(0, 0) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

which has eigenvalues $3, 1$, hence is positive definite (or you can just see this directly), so $(0, 0)$ is a local minimum.

$$Hg(1, 1) = Hg(-1, -1) = \begin{bmatrix} -3 & -3 \\ -3 & 1 \end{bmatrix}$$

If we let Q denote the associated quadratic form, then $Q(1, 0) = -3 < 0$ and $Q(0, 1) = 1 > 0$, so Q is indefinite, and $(1, 1), (-1, -1)$ are saddle points. Summing up, the critical points are $(0, 0)$ (local minimum), and $(1, 1), (-1, -1)$ (saddle points).

3. (a) (3 points) Say what it means for a set $S \subset \mathbb{R}^n$ to be a *subspace of* $\mathbb{R}^n$.

A set $S \subset \mathbf{R}^n$ is a *subspace* of $\mathbf{R}^n$ if it satisfies the following three conditions:

(i) $\mathbf{0} \in S$.

(ii) If $\mathbf{x}, \mathbf{y} \in S$, then $\mathbf{x} + \mathbf{y} \in S$.

(iii) If $\mathbf{x} \in S$ and $c \in \mathbf{R}$, then $c\mathbf{x} \in S$.

(b) (5 points) Let $C = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 | 2x + 3y - z = 0 \text{ and } x \geq 0\} \subset \mathbb{R}^3$. Is C a subspace of $\mathbb{R}^3$? Justify your answer completely.

No, C is not a subspace of $\mathbf{R}^3$, since it is not closed under scalar multiplication. For example, $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \in C$, but $(-1)\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -5 \end{bmatrix} \notin C$, because its x-component, -1, is not ≥ 0.

(c) (3 points) Give the definition of the *dimension* of a subspace $S \subset \mathbb{R}^n$.

The *dimension* of a subspace $S \subset \mathbf{R}^n$ is the size (meaning, the number of elements) of any basis of S.

(d) (5 points) Calculate, with justification, the the dimension of the subspace $X = \text{span}(\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \\ -8 \end{bmatrix})$ of $\mathbb{R}^4$? (you may take for granted that X, being the span of a collection of vectors, *is* in fact a subspace).

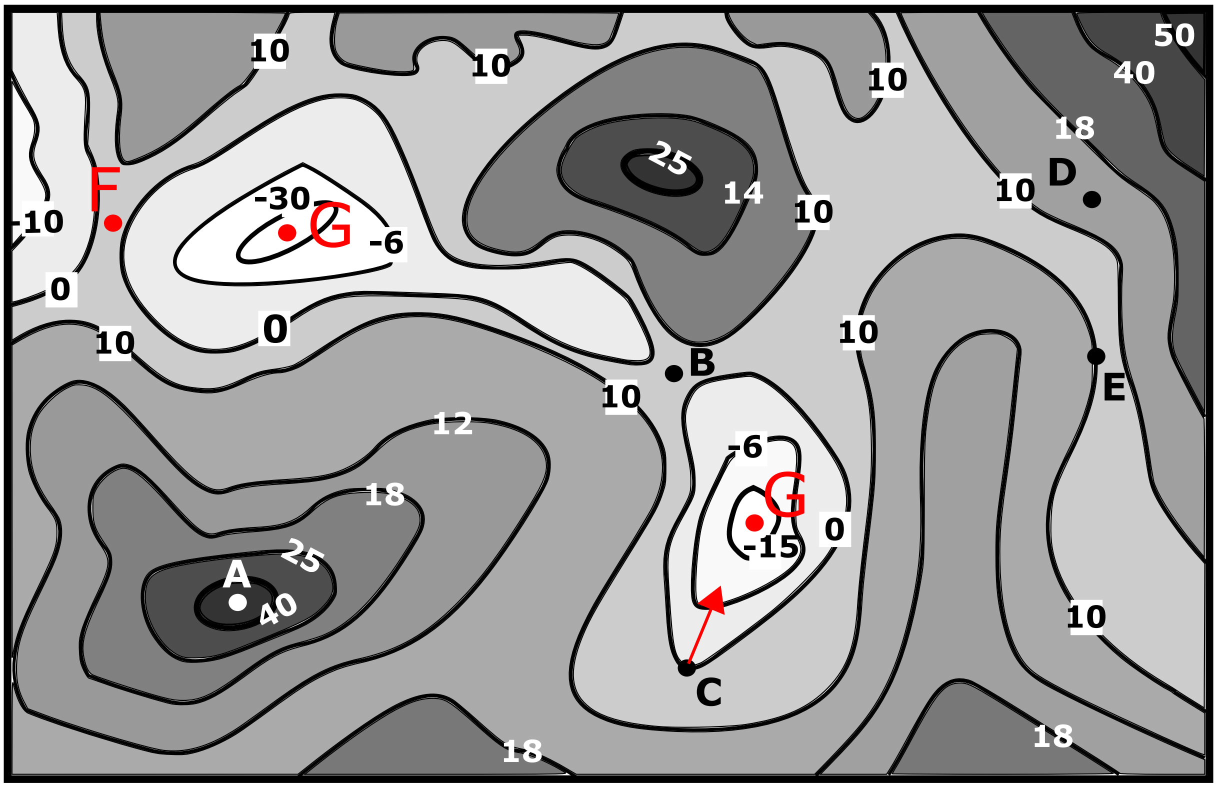
Note that $\begin{bmatrix} 2 \\ 5 \\ -1 \\ -8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$. So $X = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}\right\}$. Clearly $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$ are not collinear, so these two vectors form a basis for X, hence $\dim(X) = 2$.

(Another method for finding such a basis is as follows: note that $X = \text{span}(\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \\ -8 \end{bmatrix}) = C(A)$, where A is the matrix whose columns are these vectors

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 5 \\ 0 & 1 & -1 \\ -1 & 4 & -8 \end{bmatrix};$$

now compute that $\text{rref}(A)$ has pivots in the first two columns, so the first two columns of A form a basis for $X = C(A)$. Hence $\dim(X) = 2$).

4. *This question does not require any justification for your answers.* Below is a contour map of a function $f : \mathbb{R}^2 \to \mathbb{R}$, e.g., a collection of level curves. You may assume that f has continuous first and second derivatives, and that the scales on the x and y axes, which are parallel to the edges of the box, are the same. The numbers indicated are the heights of the various level sets, and the contour map is also (roughly) shaded with darker regions representing points where f is relatively higher, and lighter regions representating points where f is relatively lower.



(a) (2 points) Sketch, on the plot, the direction of steepest decrease of f at the point C.

The direction is drawn on the image above at point C. The key point is that this direction, which is $-\nabla f(C)$ should be *orthogonal* to the level curve containing C and pointing in the direction in which f is decreasing.

(b) (2 points) Sketch, on the plot, the approximate location of a critical point which is a saddle point which is not any of A, B, C, D, or E. Label your point "F".

There is only one possible point, which is drawn on the image above at the point F.

(c) (2 points) Sketch, on the plot, the approximate location of a critical point of f at which the Hessian of f is positive definite, which is not any of A, B, C, D, E, or F. Label your point "G".

Two possible points, both labeled G, are drawn in the picture above.

(d) (2 points) (Circle one) At the point C the value of $\frac{\partial^2 f}{\partial x^2}(C)$ is **[POSITIVE]** ZERO NEGATIVE.

(e) (2 points) (Circle one) If $\mathbf{v}$ is the unit vector in the direction of $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$, then at the point A, the directional derivative $D_{\mathbf{v}} f(A)$ is POSITIVE **[ZERO]** NEGATIVE.

(f) (2 points) (Circle one) If $\mathbf{v}$ is the unit vector in the direction of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then at the point D, the directional derivative of the partial derivative $D_{\mathbf{v}}(\frac{\partial f}{\partial x})(D)$ is **[POSITIVE]** ZERO NEGATIVE.

(g) (2 points) (Circle one) At the point E the value of $\frac{\partial f}{\partial y}(E)$ is POSITIVE **[ZERO]** NEGATIVE

5. (10 points) You would like to understand the approximate relationship between the *number of floors a given building has*, x and the building's *height* y in meters. You examine the 3 buildings closest to you and find the number of floors x_i and the height y_i of each building to be:

- Building 1: $(x_1, y_1) = (1, 5)$
- Building 2: $(x_2, y_2) = (3, 10)$,
- Building 3: $(x_3, y_3) = (6, 20)$.

To try to understand the rough relationship between x and y, you would like to find a *linear regression* for the data you have collected , meaning a line $y = mx + b$ which *best fits* the data points you have collected, in the following sense: to a given line $y = mx + b$ which is determined by a slope m and a y-intercept b, you can associate an *error function* which measures the *deviation* from this line and the data points above, meaning its failure to match the points. In this case, the function is given in terms of m and b as follows:

$$E(m, b) = \sum_{i=1}^{3}(y_i - (mx_i + b))^2 = (5 - (m \cdot 1 + b))^2 + (10 - (m \cdot 3 + b))^2 + (20 - (m \cdot 6 + b))^2.$$

You may take for granted that there is a unique (m_0, b_0) that minimizes the function $E(m, b)$; the associated line $y = m_0x + b_0$ is called the *best fit line* to the data points above.

Find, using calculus, the best fit line to the data points above. *Note:* there is no need to simplify m_0 and b_0 once you have found them. **Hint**: Since you are told a global minimum exists, you know the global minimum must be a local minimum and in particular a critical point of E. If E has only one critical point, there is therefore no need to perform a second derivative test.

By the hint, (m_0, b_0) is one of the critical points of E. We calculate that the gradient of E as a function of m and b is:

$$\nabla E(m, b) = \begin{bmatrix} E_m(m, b) \\ E_b(m, b) \end{bmatrix} = \begin{bmatrix} -2(5 - m - b) - 6(10 - 3m - b) - 12(20 - 6m - b)) \\ -2(5 - m - b) - 2(10 - 3m - b) - 2(20 - 6m - b) \end{bmatrix}.$$

Critical points of E are precisely the points m, b where $\nabla E(m, b) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, meaning points (m, b) solving the following system of equations:

$$-2(5 - m - b) - 6(10 - 3m - b) - 12(20 - 6m - b)) = 0$$
$$-2(5 - m - b) - 2(10 - 3m - b) - 2(20 - 6m - b) = 0.$$

Simplifying this, we see that m and b satisfy the system

$$92m + 20b = 310$$
$$20m + 6b = 70.$$

Solving this system (for instance by subtracting 10 times the second equation from 3 times the first to eliminate b, solve for m, and substituting back into either equation to solve for b), we obtain one solution, and hence just one critical point:

$$(m_0, b_0) = (\frac{115}{38}, \frac{30}{19})$$

Since (m_0, b_0) is the only solution to this system and hence the only critical point of E, we conclude by the hint that it must be the global minimum. Therefore the best fit line is

$$y = \frac{115}{38}x + \frac{30}{19}.$$

As mentioned in the problem statement, complete simplification was not required to obtain full credit.

6. Consider the following quadratic form

$$q(x, y, z) = 2x^2 - 3y^2 - z^2 - 4yz.$$

(a) (4 points) Write down a symmetric matrix A whose associated quadratic form equals q; that is, a matrix with $A^T = A$ so that $q(x, y, z) = Q_A(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & -2 \\ 0 & -2 & -1 \end{bmatrix}.$$

(b) (6 points) Determine with justification the definiteness of the quadratic form $q(x, y, z)$.

Note that $q(1, 0, 0) = 2 > 0$ and $q(0, 1, 0) = -3 < 0$. Since there is at least one point where q is positive, and at least one point where q is negative, we conclude that q must be *indefinite*.

(one could also compute the eigenvalues of A via finding the eigenvalues of A, which are zeroes of the characteristic polynomial. A computation reveals that the eigenvalues are $2, -2 - \sqrt{5}, -2 + \sqrt{5}$. since at least one eigenvalue of A is positive and at least one eigenvalue of A is negative, Q_A must be indefinite).

(c) (6 points) Set up, but do not solve, system of equations in x, y, z and maybe other unknown variables, which would find the local extrema (along with local saddle points) of $q(x, y, z)$ restricted to the region $S = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x^2 - 2y^2 = 1, y^2 + z^2 = 4\}$. For full credit, you should write your system as a list of equations with as many equations as there are unknowns (so if x, y, and z are the only unknowns, you should write three equations).

We note that S is the intersection of level sets $g_1^{-1}(1) \cap g_2^{-1}(4)$ where $g_1(x, y, z) = x^2 - 2y^2$ and $g_2(x, y, z) = y^2 + z^2$. Finding the local extrema of f restricted to S is therefore a problem about finding local extrema with multiple constraints; so we will apply the Lagrange Multipliers theorem with multiple constraints. First, we compute the gradient vectors of q, g_1, and g_2:

$$\nabla(q)(x, y, z) = \begin{bmatrix} 4x \\ -6y - 4z \\ -2z - 4y \end{bmatrix}, \ \nabla g_1(x, y, z) = \begin{bmatrix} 2x \\ -4y \\ 0 \end{bmatrix}, \ \nabla g_2(x, y, z) = \begin{bmatrix} 0 \\ 2y \\ 2z \end{bmatrix}.$$

First, note that $\nabla g_1(x, y, z)$ and $\nabla g_2(x, y, z)$ are always linearly independent on S. Indeed, both vectors are always non-zero: $\nabla g_1 \neq \mathbf{0}$ on S because x is never zero on S; otherwise $g_1(x, y, z)$ would be ≤ 0, and $\nabla g_2(x, y, z) \neq \mathbf{0}$ on S, because it only $= \mathbf{0}$ when $y = z = 0$ which does not lie on S. Also, the vectors are never collinear because $\nabla g_1(x, y, z)$ always has non-zero first component on S (by prior reasoning, since we argued $x \neq 0$ on S), and $\nabla g_2(x, y, z)$ always has zero first component.

Since $\nabla g_1(x, y, z)$ and $\nabla g_2(x, y, z)$ are linearly independent on S, the theorem of Lagrange multipliers with multiple constraints tell us that at a local extremum (or local saddle point) $\mathbf{x}$ of $q|_S$, for some scalars λ_1, λ_2

$$\nabla q(\mathbf{x}) = \lambda_1 g_1(\mathbf{x}) + \lambda_2 g_2(\mathbf{x}).$$

to find points on S where the above is satisfied, we solve the above vector-based equation (written as a system of equations in $x, y, z, \lambda_1, \lambda_2$) along with the two constraint equations $g_1(\mathbf{x}) = 1$, $g_2(\mathbf{x}) = 4$ which force the solution to lie on S. Writing the above vector equation as a system of equations along with the two constraint equations, we obtain a system of 5 equations in 5 unknowns $x, y, z, \lambda_1, \lambda_2$:

$$\begin{cases} 4x & = \lambda_1(2x) + \lambda_2(0) \\ -6y - 4z & = \lambda_1(-4y) + \lambda_2(2y) \\ -2z - 47 & = \lambda_1(0) + \lambda_2(2z) \\ x^2 - 2y^2 & = 1 \\ y^2 + z^2 & = 4 \end{cases}$$

7. (a) (8 points) Compute the second order Taylor approximation to the function

$$f(x, y) = \ln(x^2 + y^2)$$

at $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Express your answer as a polynomial in the form $a + b(x - \frac{1}{\sqrt{2}}) + c(y - \frac{1}{\sqrt{2}}) + d(x - \frac{1}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}) + e(x - \frac{1}{\sqrt{2}})^2 + f(y - \frac{1}{\sqrt{2}})^2$.

The formula for the second order Taylor approximation for a real valued function f near a point $\mathbf{a}$ is given by

$$\begin{aligned} T_2(\mathbf{x}) &= f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2}\mathbf{x}^T Hf(\mathbf{a})\mathbf{x} \\ &= f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}\mathbf{x} \cdot (Hf(\mathbf{a})\mathbf{x}); \end{aligned}$$

in this case $\mathbf{a} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. One can compute that

$$\begin{aligned} f(\mathbf{a}) &= \ln(1) = 0; \\ \nabla f(\mathbf{a}) &= \begin{bmatrix} \frac{2x}{x^2+y^2} \\ \frac{2y}{x^2+y^2} \end{bmatrix} \Big|_{(x,y)=(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \\ Hf(\mathbf{a}) &= \begin{bmatrix} \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} & \frac{-4xy}{(x^2+y^2)^2} \\ \frac{-4xy}{(x^2+y^2)^2} & \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} \end{bmatrix} \Big|_{(x,y)=(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \end{aligned}$$

Substituting this into the definition of T_2 and simplifying, we obtain

$$\begin{aligned} T_2(x, y) &= 0 + \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} x - \frac{1}{\sqrt{2}} \\ y - \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x - \frac{1}{\sqrt{2}} \\ y - \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \left(\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x - \frac{1}{\sqrt{2}} \\ y - \frac{1}{\sqrt{2}} \end{bmatrix} \right) \\ &= \sqrt{2}(x - \frac{1}{\sqrt{2}}) + \sqrt{2}(y - \frac{1}{\sqrt{2}}) - 2(x - \frac{1}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}). \end{aligned}$$

(b) (4 points) Use the result from part (a) to estimate the value of $\ln(1.02) = f(\frac{1}{\sqrt{2}} + 0.1, \frac{1}{\sqrt{2}} - 0.1)$.

If $(x, y) = (\frac{1}{\sqrt{2}} + 0.1, \frac{1}{\sqrt{2}} - 0.1)$, then $x - \frac{1}{\sqrt{2}} = 0.1$ andc $y - \frac{1}{\sqrt{2}} = -0.1$. We plug these two values into the formula above to obtain

$$f(\frac{1}{\sqrt{2}} + 0.1, \frac{1}{\sqrt{2}} - 0.1) \approx T_2(\frac{1}{\sqrt{2}} + 0.1, \frac{1}{\sqrt{2}} - 0.1) = \sqrt{2}(0.1) + \sqrt{2}(-0.1) - 2(0.1)(-0.1) = 0.02.$$

8. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 5 & -3 & 2 \\ 2 & a & 1 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, for some a.

(a) (6 points) Under what conditions if any on a is $\mathbf{b} \in C(A)$? Justify your work.

We write down an augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 5 & -3 & 2 & 2 \\ 2 & a & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -8 & 2 & -3 \\ 0 & a-2 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1/4 & 3/8 \\ 0 & a-2 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 5/8 \\ 0 & 1 & -1/4 & 3/8 \\ 0 & 0 & \frac{a}{4}+\frac{1}{2} & -\frac{3}{8}a-\frac{1}{4} \end{array}\right]$$

If $\frac{a}{4} + \frac{1}{2} = 0$, i.e. if $a = -2$, then one checks directly that $-\frac{3}{8}a - \frac{1}{4} \neq 0$, so we obtain an inconsistent equation, hence $\mathbf{b} \notin C(A)$. If $a \neq -2$, then the last column is a pivot column, hence we can row reduce to the identity and obtain a solution to the above system, so $\mathbf{b} \in C(A)$. So the condition for $\mathbf{b} \in C(A)$ is $a \neq -2$.

(b) (6 points) Under what conditions on a are there

- no solutions,
- exactly one solution,
- many solutions

to $A\mathbf{x} = \left[\begin{smallmatrix} 0 \\ 2 \\ 1 \end{smallmatrix}\right]$? (for each part your answer may be of the form $a = c$, $a \neq c$, $a \geq c$ or *never*). Justify your work.

Since $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ is the last column of A, it lies in $C(A)$, so $A\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ always has a solution, namely, $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Now the set of solutions is a translate of $N(A)$, so there is only one solution precisely when $N(A) = \{\mathbf{0}\}$, i.e. when $\det(A) \neq 0$. From our row reduction calculation in part (a), $\det(A) = -8(1)(1)(\frac{a}{4} + \frac{1}{2}) = -2a - 4$. This is 0 precisely when $a = -2$. Thus, there is one solution if $a \neq -2$, and there are many solutions if $a = -2$.

9. (12 points) (1 point for each sub-part) *This question does not require you to show your work.* In each of the following parts, you will be given a function f and a region $E \in \mathbb{R}^n$. Say first (i) whether E is closed or not closed, (ii) whether E is bounded or not bounded, and (iii) whether f attains an absolute minimum on the region E.

(a) $E = \{(x, y, z) \mid -1 \leq x \leq 2, 1 \leq y \leq 3, 5 \leq z \leq 7\}$. $f(x, y, z) = xyz - x^2ye^{xy}$.

i. (circle one) E is **[CLOSED]** NOT CLOSED.
ii. (circle one) E is **[BOUNDED]** NOT BOUNDED.
iii. (circle one) The restriction of f to E **[ATTAINS]** DOES NOT ATTAIN an absolute minimum on the region E.

E is closed (defined by weak inequalities) and bounded, since by definition x, y, z are restricted to certain bounded intervals, so by the Extreme Value Theorem, since f is continuous, the restriction of f to E attains a minimum.

(b) $E = \{(x, y, z, w) \mid x < 1, y < 2, (w + z) < 2, \} \subset \mathbb{R}^4$, $f(x, y, z, w) = 6 - w - z$.

i. (circle one) E is CLOSED **[NOT CLOSED]**.
ii. (circle one) E is BOUNDED **[NOT BOUNDED]**.
iii. (circle one) The restriction of f to E ATTAINS **[DOES NOT ATTAIN]** an absolute minimum on the region E.

E is not closed, since e.g. it doesn't contain the boundary point $(1, 2, 1, 1)$. It is not bounded, since e.g. x can be arbitrarily negative. The restriction of f to E does not attain a minimum, since that would correspond to a maximum for the function $w + z$, and $w + z$ gets arbitrarily close to, but never hits, the value 2 on E.

(c) $E = \{(x, y, z) \mid 2x - y + z = 50\}$, $f(x, y, z) = x^2 + y^2 + z^2$.

i. (circle one) E is **[CLOSED]** NOT CLOSED.
ii. (circle one) E is BOUNDED **[NOT BOUNDED]**.
iii. (circle one) The restriction of f to E **[ATTAINS]** DOES NOT ATTAIN an absolute minimum on the region E.

E is closed (defined by an equality), but not bounded, since for any values of x, y, one can find a corresponding z value such that $(x, y, z) \in E$, i.e. $z = 50 - 2x + y$. But the restriction of f to E does attain a minimum, since f is the distance squared from (x, y, z) to the origin, and any closed set has a point closest to the origin, (we have shown this in a homework problem, for instance).

(d) $E = \{(x, y) \mid x^2 - 4y^2 \leq 5,\ |y| \leq 10\}$, $f(x, y) = 2x + 3y$.

i. (circle one) E is **[CLOSED]** NOT CLOSED.
ii. (circle one) E is **[BOUNDED]** NOT BOUNDED.
iii. (circle one) The restriction of f to E **[ATTAINS]** DOES NOT ATTAIN an absolute minimum on the region E.

E is closed (defined by weak inequalities) and bounded: The condition $|y| \leq 10$ ensures that y is bounded, and the condition $x^2 \leq 5 + 4y^2$ then ensures that x^2, hence x, is bounded. By the Extreme Value Theorem, therefore, since f is continuous, the restriction of f to E attains an absolute minimum.

10. (14 points) Let $f(x, y) = 2x^2 - x + 3y^2 + 4$. Find the absolute minimum and maximum value attained by f when restricted to the *disc*

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

You may assume that a global minimum and maximum of $f|_D$ exist.

We first find the critical points of f on the interior by setting $\nabla f(x, y) = \begin{bmatrix} 4x - 1 \\ 6y \end{bmatrix}$ equal to $\mathbf{0}$. This yields $(x, y) = (\frac{1}{4}, 0)$. This point is indeed in the interior of D, since $(\frac{1}{4})^2 + 0^2 < 1$.

Now we use Lagrange multipliers to optimize f on the boundary $x^2 + y^2 = 1$ of D. If we let $g(x, y) = x^2 + y^2$, then $\nabla g(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. If this is $\mathbf{0}$, then $(x, y) = (0, 0)$, which does not satisfy the constraint $x^2 + y^2 = 1$. So $\nabla g \neq \mathbf{0}$ on the boundary of D. Therefore, the critical points of f restricted to the boundary are those for which $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbf{R}$. That is,

$$\begin{bmatrix} 4x - 1 \\ 6y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

The second coordinate yields that either $y = 0$ or $\lambda = 3$. If $y = 0$, then since $x^2 + y^2 = 1$, we get $x = \pm 1$, so this yields the two points $(\pm 1, 0)$. If $\lambda = 3$, then looking at the first coordinate yields $4x - 1 = 6x \Longrightarrow x = -\frac{1}{2}$. Then the constraint yields $y = \pm\frac{\sqrt{3}}{2}$. So we get the two points $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. We now compute the value of f at the five points that we have found, and we see that $f|_D$ attains a minimum value of $\frac{31}{8}$ at $(\frac{1}{4}, 0)$ and a maximum value of $\frac{29}{4}$ at $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$.

11. Let $S = g^{-1}(5) \subset \mathbb{R}^3$ be the level set of a differentiable function g. Suppose you are told that the point $\mathbf{a} = (1, -1, 2)$ lies on surface S and morevoer that at $(1, -1, 2)$, there is a tangent plane to S which is given parametrically as

$$P = \{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \}.$$

(a) (5 points) Find an equation for the plane P in the form $ax + by + cz = d$.

We are told two linearly independent vectors which are parallel to P, namely

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore we can find a vector normal to P by taking the *cross product* of $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

We are also given a point on the plane, namely $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. The equation for a plane P containing point $\mathbf{a}$ with normal vector $\mathbf{n}$ is given by $\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0$ which one can write as $n_1(x - a_1) + n_2(x - a_2) + n_3(x - a_3) = 0$. In this case, this gives us the equation

$$1(x - 1) + 0(y + 1) - 1(z - 2) = 0,$$

which simplifies to

$$x - z = -1.$$

(b) (6 points) Now, let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function given by $f(x, y, z) = \frac{3}{4}x^2 - xe^{y+1} + bzy$ for some scalar $b \in \mathbb{R}$. Show that it is not possible for the restriction $f|_S$ of f to S to have a local extremum at $(1, -1, 2)$ unless $b = \frac{1}{2}$. *Hint*: think about the what is required for $f|_S$ to have a local extremum at $(1, -1, 2)$ geometrically.

The theorem of Lagrange multipliers tells is that, as long as $\nabla g(1, -1, 2) \neq 0$, if $f|_{g^{-1}(5)}$ were to have a local extremum at $(1, -1, 2)$, then $\nabla f(1, -1, 2)$ should be collinear to $\nabla g(1, -1, 2)$. We know that g is differentiable and that there is a tangent plane to $g^{-1}(5)$ with normal vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$; meaning that we can assume $\nabla g(1, -1, 2) \neq \mathbf{0}$ and is collinear with $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (remember that $\nabla(g)(\mathbf{a})$ is always orthogonal to the tangent plane to $S = g^{-1}(c)$ at $\mathbf{a} \in S$). In total, we conclude $\nabla f(1, -1, 2)$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ must be collinear. (Note: many students said $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ needs to equal $\nabla g(1, -1, 2)$, which is *not necessarily* the case; this resulted in a partial deduction. Insead, we only know that those two vectors are collinear).

Now, compute that

$$\nabla f(1, -1, 2) = \begin{bmatrix} \frac{3}{2}x - e^{y+1} \\ -xe^{y+1} + bz \\ by \end{bmatrix} |_{(x,y,z)=(1,-1,2)} = \begin{bmatrix} \frac{1}{2} \\ -1 + 2b \\ -b \end{bmatrix}.$$

For this to be collinear with $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, the second component of $\nabla f(1, -1, 2)$ should at the very least be zero, meaning b must equal $\frac{1}{2}$. We then check that when $b = \frac{1}{2}$, $\nabla f(1, -1, 2) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$, which is indeed collinear with (and indeed equals $-1/2$ times) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Note: Another way to say the first paragraph more briefly is this: to have a local extremum of $f|_S$ at $(1,-1,2)$, $\nabla f(1,-1,2)$ needs to be orthogonal to the tangent plane to S at $(1,-1,2)$, but $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is a normal vector to this tangent plane, so $\nabla f(1,-1,2)$ is collinear with $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(c) (5 points) Now, let $\mathbf{r} : [0,1] \to \mathbb{R}^3$ be a differentiable parametric curve. Suppose that $\mathbf{r}(\frac{1}{2}) = (1,-1,2)$, and that $\mathbf{r}'(\frac{1}{2})(= D\mathbf{r}(\frac{1}{2})) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Under what conditions, if any, on b, does the composition $f \circ \mathbf{r}$ have a critical point at $\frac{1}{2}$?

Let $g = f \circ \mathbf{r}$. By the chain rule (and using the fact that $Df(\mathbf{a}) = \nabla f(\mathbf{a})^T$),

$$\begin{aligned} g'(\frac{1}{2}) &= Df(\mathbf{r}(\frac{1}{2}))D\mathbf{r}(\frac{1}{2}) \\ &= \nabla f(1,-1,2) \cdot \mathbf{r}'(\frac{1}{2}) \\ &= \begin{bmatrix} \frac{1}{2} \\ 1+2b \\ -b \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \\ &= 1 + (-1+2b) - b \\ &= b. \end{aligned}$$

g has a critical point at $\frac{1}{2}$ when $g'(\frac{1}{2}) = 0$, which happens precisely (since $g'(1/2) = b$) when $b = 0$.

12. Let A be a 2×2 matrix. Suppose you are not told A, but instead are given that

$$\text{rref}(2I - A) = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}, \quad \text{rref}(5I - A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{rref}(-I - A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

(a) (6 points) Calculate, with justification, a basis of $\mathbb{R}^2$ consisting of eigenvectors of A, and say what their corresponding eigenvalues are.

Since rref$(2I - A)$ and rref$(-I - A)$ are not the identity, 2 and -1 are eigenvalues (hence all of the eigenvalues) of A. To compute the 2-eigenspace, we compute $N(2I - A) = N(\text{rref}(2\text{I} - \text{A})) =$ N$\left(\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}\right)$. This corresponds to the equation $x - \frac{1}{3}y = 0$, so the 2-eigenspace is span$\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$. Similarly, the -1-eigenspace is span$\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$. Therefore, an eigenbasis for A is $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$, with corresponding eigenvalues 2 and -1, respectively.

(b) (6 points) Determine, using any method of your choice, the matrix of A.

The matrix of the linear transformation T associated to A with respect to the basis $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ is $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. To find A, the matrix of T in standard coordinates, we use the change of basis matrix $C = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$. We first compute C^{-1} by augmenting the identity and row reducing:

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 7 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{3}{7} & \frac{1}{7} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & -\frac{3}{7} & \frac{1}{7} \end{array}\right]$$

Now $A = CBC^{-1} = \begin{bmatrix} -\frac{4}{7} & \frac{6}{7} \\ \frac{9}{7} & \frac{11}{7} \end{bmatrix}$.

13. (14 points) (2 points each) Each of the statements below is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). For each part, decide which and circle the appropriate choice; you *do not* need to justify your answers.

(a) The linear transformation $Proj_L : \mathbb{R}^2 \to \mathbb{R}^2$ given by projecting to a line L is invertible. T **[F]** MAYBE

FALSE. Projecting to a line sends everything orthogonal to that line to $\mathbf{0}$, hence cannot be invertible.

(b) If a 3×3 matrix A has characteristic polynomial $p(\lambda) = \lambda(\lambda - 1)(\lambda + 3)$, then A is diagonalizable. **[T]** F MAYBE

A is a 3×3 matrix with 3 distinct eigenvalues $(0, 1, -3)$, hence is diagonalizable.

(c) Let A be an $n \times n$ matrix whose first two rows are equal. Then A is invertible. T **[F]** MAYBE

If two of the rows are equal, then the rows are not linearly independent, and in particular rref(A) has a row with all zeroes, so must have fewer than n pivots. Thus, A is not invertible.

(d) If A is a 2×2 matrix, then $\det(A) = \det(\text{rref}(A))$. T F **[MAYBE]**

This might be true, e.g. if A is already in reduced row echelon form. But if $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, for example, then $\det(A) = 2$, while $\text{rref}(A) = I$, which has determinant 1.

(e) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function with continuous first and second partials. If $\mathbf{a} \in \mathbb{R}^3$ is a critical point of f, and the Hessian matrix at $\mathbf{a}$, $Hf(\mathbf{a})$, has eigenvalues 3, 2, and 0, then $\mathbf{a}$ is a local maximum of f. T **[F]** MAYBE

By the classification of quadratic forms, we know that $Q_{Hf(\mathbf{a})}$ is *positive semi-definite.* As discussed in class, the second derivative test, while inconclusive in this case, does exclude some possibilities: $\mathbf{a}$ *cannot* be a local maximum of f, it can only be a local minimum or a saddle point.

Here is one way to see this: Since the Hessian has some positive eigenvalues, there are lines (i.e. the span of a corresponding eigenvector) such that the restriction of f to that line, thought of as a single-variable function, has positive second derivative. By the second derivative test from single-variable calculus, therefore, the restriction of f to that line has a local minimum at $\mathbf{a}$ (and, what's more, not a local maximum, because f is not constant). Since this restriction doesn't even have a local maximum at $\mathbf{a}$, there's no way that f does.

(f) If A is a 4×2 matrix and B is a 2×4 matrix, then BA is the identity matrix. T F **[MAYBE]**

This can be false, e.g. if A, B are the zero matrices. It can also be true, e.g. if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(g) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function whose partial derivatives $\frac{\partial f}{\partial x_i}$ are all defined at $\mathbf{a} \in \mathbb{R}^2$. Then f is differentiable at $\mathbf{a}$. **[T]** F MAYBE

Note: There was a typo on this problem on the exam; the problem said that $\mathbf{a} \in \mathbf{R}^4$ *(even though* $f : \mathbb{R}^2 \to \mathbb{R}$*). Due to this confusion, we gave credit to all solutions to this problem.*

With the above corrected problem, though, the answer is MAYBE: It can be true, by taking f to be any differentiable function. It can also be false, e.g. by taking $f(x, y) = \sqrt{xy}$, and $\mathbf{a} = \mathbf{0}$.

Then

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

and similarly $f_y(0,0) = 0$. But f is not differentiable at $(0,0)$. Indeed, if it were then $\lim_{(x,y)\to(0,0)} \frac{|f(x,y)-L(x,y)|}{\sqrt{x^2+y^2}} = 0$, where $L(x,y)$ is the linearization of f at $(0,0)$. By the above calculations, $L(x,y) = 0$, so for f to be differentiable at $(0,0)$, we would ned

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{xy}}{\sqrt{x^2+y^2}} = 0$$

But this is false. For example, if we let $(x,y) \to (0,0)$ along the line $y = x$, then we get

$$\lim_{x\to 0} \frac{|x|}{|x|\sqrt{2}} = \frac{1}{\sqrt{2}}$$

So f is not differentiable at $(0,0)$.

(Recall that the definition of differentiability of f at $\mathbf{a}$ asks not just that the partial derivatives at $\mathbf{a}$ exist but that f should have a *best linear approximation* at $\mathbf{a}$. The above example is an instance of when the formal is satisfied but not the latter)