Math 51 Midterm 1 — July 6, 2016

Name: ______________________________ **SUID#:** ______________

Circle your section:	
Section 01 (1:30-2:50PM)	Section 02 (3:00-4:20PM)

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 7 numbered pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 1 hour and 20 minutes.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- Paper not provided by teaching staff is prohibited. If you need extra room for your answers, use the back side of the question page or other extra space provided at the front of this packet, and clearly indicate that your answer continues there. Do not unstaple or detach pages from this exam.
- Please show your work unless otherwise indicated. *Do not make multiple guesses: if there are multiple answers given to a question that only asks for one answer, your grade will be the minimum of the possible scores you would have received.*
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner. You have only until **one week after your graded midterm is made available**, to resubmit your exam for any regrade considerations; consult one of your CAs for the exact details of the submission process.
- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the Stanford's honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	**Total**
Points:	11	15	10	15	12	12	75
Score:							

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1. (a) (6 points) Find an equation (of the form $ax + by + cz = d$) for the plane P in $\mathbb{R}^3$ passing through the points $(1, 2, 1)$, $(2, 1, 0)$, and $(0, 0, 1)$.

   (b) (5 points) Can you express this plane as the null space of a matrix? That is, is there a matrix A with $N(A) = P$? Why or why not?

2. Suppose A is a 4×2 matrix whose column space $C(A)$ admits the following description:

$$C(A) = \{\mathbf{b} \in \mathbb{R}^4 | \mathbf{b} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = 0 \text{ and } \mathbf{b} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0\}$$

(a) (5 points) Find with justification a basis for the the column space of A (hint: if $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \in \mathbb{R}^4$, express the condition for $\mathbf{b} \in C(A)$ as equations satisfied by the b_i).

(b) (5 points) Are there zero, one or many solutions to the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$? Explain.

(c) (5 points) Are there zero, one, or many solutions to the equation $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$? Explain.

3. (a) (4 points) Say what it means for a list of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ to be *linearly independent.*

(b) (6 points) Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in $\mathbb{R}^n$. Is the collection $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - 2\mathbf{v}, \mathbf{w} + \mathbf{u} - \mathbf{v}\}$ linearly independent or linearly dependent? Show your work.

4. (a) (4 points) Say what it means for a function $T : \mathbb{R}^n \to \mathbb{R}^m$ to be a *linear transformation.*

(b) (5 points) Let $\mathbf{v} \in \mathbb{R}^n$ be any non-zero vector. Recall that on homework, we defined the *projection onto the vector* $\mathbf{v}$, $Proj_{\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}^n$, by the following formula: $\text{Proj}_{\mathbf{v}}(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$. It is a fact mentioned in class that $Proj_{\mathbf{v}} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a *linear transformation.*

Now, let $P \subset \mathbb{R}^n$ be a plane through the origin, described as $P = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ for a pair of perpendicular non-zero vectors $\mathbf{v}_1, \mathbf{v}_2$. We can define the *projection onto the plane* P

$$\text{Proj}_P : \mathbb{R}^n \to \mathbb{R}^n$$

via the following formula:

$$\text{Proj}_P(\mathbf{x}) := \text{Proj}_{\mathbf{v}_1}(\mathbf{x}) + \text{Proj}_{\mathbf{v}_2}(\mathbf{x}).$$

Show directly from definitions that $Proj_P$ is a linear transformation. You may use the fact that $Proj_{\mathbf{v}}$ is a linear transformation for any $\mathbf{v}$ without justification (though you should state how you are using it).

(c) (6 points) Now let $n = 3$ (so we are working in $\mathbb{R}^3$), and let $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, and $P = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$. Compute the matrix associated to Proj_P in this case.

You may use the following fact if it is helpful: Both $\mathbf{v}_1$ and $\mathbf{v}_2$ are *unit-length vectors*, meaning $||\mathbf{v}_1|| = ||\mathbf{v}_2|| = 1$. (Note that when $\mathbf{m} \in \mathbb{R}^n$ is unit length, the formulae for projection onto $\mathbf{m}$ simplifies to: $\text{Proj}_{\mathbf{m}}(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{m})\mathbf{m}$).

5. (12 points) Find, showing all your work and using any method of your choice, all solutions (in vector form) to the system of equations:

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & 1 & -8 \\ 5 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ -3 \end{bmatrix}.$$

Describe your solutions as a point, line, or plane.

6. (12 points) (2 points each) Each of the statements below is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). For each part, decide which and circle the appropriate choice; you *do not* need to justify your answers.

   (a) Let A be a 3×4 matrix. If $\dim C(rref(A)) = 2$, then $\dim C(A) = 2$. T F MAYBE

   (b) If $P \subset \mathbb{R}^n$ is a plane going through the origin, and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in P, then P is spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. T F MAYBE

   (c) If the reduced row echelon form of the augmented matrix $[A|\mathbf{b}]$ has two free variables, then there are many solutions to the associated system. T F MAYBE

   (d) If A is a 2×3 matrix, then $N(A) \neq \{\mathbf{0}\}$. T F MAYBE

   (e) If A is a 3×2 matrix, then $N(A) \neq \{\mathbf{0}\}$. T F MAYBE

   (f) Suppose V is a subspace in $\mathbb{R}^n$ of dimension d, and a collection of d vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ spans V. Then this collection is also linearly independent. T F MAYBE

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