# Solutions to Math 51 Midterm 1 — July 6, 2016

1. (a) (6 points) Find an equation (of the form $ax + by + cz = d$) for the plane $P$ in $\mathbb{R}^3$ passing through the points $(1, 2, 1)$, $(2, 1, 0)$, and $(0, 0, 1)$.

We first compute two non-collinear vectors parallel to the plane:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

A normal vector is found by taking the cross product of any two non-collinear vectors which are parallel to $P$; taking the cross product of the above two vectors, we get the following normal vector:

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

The equation for the plane $P$ then is

$$\mathbf{n} \cdot \left( \mathbf{x} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \mathbf{0}$$

which is equivalent to $2x - y + 3z = 3$.

(b) (5 points) Can you express this plane as the null space of a matrix? That is, is there a matrix $A$ with $N(A) = P$? Why or why not?

No. $\mathbf{0}$ is not in the plane because $2 \cdot 0 - 0 + 3 \cdot 0 \neq 3$. Therefore the plane cannot be the null space of a matrix.

2. Suppose $A$ is a $4 \times 2$ matrix whose column space $C(A)$ admits the following description:

$$C(A) = \{\mathbf{b} \in \mathbb{R}^4 | \mathbf{b} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = 0 \text{ and } \mathbf{b} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0\}$$

(a) (5 points) Find with justification a basis for the the column space of $A$ (hint: if $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \in \mathbb{R}^4$, express the condition for $\mathbf{b} \in C(A)$ as equations satisfied by the $b_i$).

The conditions for $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \in C(A)$ as described in the question above can be described as the following system of equations:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = b_1 + 2b_2 - b_4 = 0 \tag{1}$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = b_2 + b_4 = 0. \tag{2}$$

The question is asking for us to find a basis for the set of all solutions (in vector form) to this pair of equations; this will give us a basis for the subspace $C(A)$. The augmented matrix of this system is $\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$. By applying one row operation (subtracting twice the first column from the second column), we can put this augmented matrix into rref:

$$\text{rref}(\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]. \tag{3}$$

This rref matrix has $b_1$ and $b_2$ as pivot variables and $b_3$ and $b_4$ as free variables. (Note that since there are two free variables, there should be two vectors in the basis). Writing out the equations, we get

$$\begin{aligned} b_1 - 3b_4 &= 0 \\ b_2 + b_4 &= 0. \end{aligned}$$

Solving for the pivots in terms of frees, we get that

$$\begin{aligned} b_1 &= 3b_4 \\ b_2 &= -b_4. \end{aligned}$$

Writing the solutions in vector form (with the free variables arbitrary elements of $\mathbb{R}$), the solutions can be described as

$$\begin{aligned} C(A) &= \{\begin{bmatrix} 3b_4 \\ -b_4 \\ b_3 \\ b_4 \end{bmatrix} | b_3, b_4 \in \mathbb{R}\} \\ &= \{b_3 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix} + b_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} | b_3, b_4 \in \mathbb{R}\} \\ &= \text{span}(\begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}). \end{aligned}$$

The vectors produced this way which span our subspace $C(A)$ are linearly independent (this is a general fact, or by inspection in this case, since they are not collinear), and therefore are a basis for $C(A)$. So, **a basis for $C(A)$ is $\{\begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\}$.**

(b) (5 points) Are there zero, one or many solutions to the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$? Explain.

The equation $A\mathbf{x} = \mathbf{b}$ has solutions precisely when $\mathbf{b} \in C(A)$. The first thing to check is, is this particular $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ in $C(A)$? We need to check whether $\mathbf{b}$ satisfies the conditions of being in $C(A)$ described in the problem statement; in particular note that

$$\mathbf{b} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = 1 + 6 = 7 \neq 0, \tag{4}$$

so the conditions are *not* satisfied, and $\mathbf{b} \notin C(A)$. This, **there are no solutions to** $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$.

(It would be sufficient also to note that for this $\mathbf{b}$, $\mathbf{b} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 3 \neq 0$).

(c) (5 points) Are there zero, one, or many solutions to the equation $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$? Explain.

First, note that $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is in $C(A)$, which one can deduce either by directly verifying that $\mathbf{b} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \mathbf{b} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0$, or by noting that $\mathbf{b}$ is one of the basis vectors of $C(A)$ computed in part (a). So there is at least one solution.

When there are solutions to $A\mathbf{x} = \mathbf{b}$, the question of whether there are exactly one solution or many solutions is a question about the *null space* of $A$: Recall that if $\mathbf{x}_p$ is one solution to $A\mathbf{x} = \mathbf{b}$, then the set of all solutions is the set $\{\mathbf{x}_p + \mathbf{w} | \mathbf{w} \in N(A)\}$, which is a translation of $N(A)$ by $\mathbf{x}_p$. So if $N(A) = \{\mathbf{0}\}$ there is exactly one solution and if $N(A) \neq \{\mathbf{0}\}$ there are many solutions.

Since we are not given the matrix $A$, we have to reason about $N(A)$ indirectly, using the information we are given. In this case, we have determined from part (a) that $\dim C(A) = 2$, so in particular since there are only two columns of $A$, these columns must be linearly independent. Therefore, by a result in the book, $N(A) = \{\mathbf{0}\}$, and **there is only one solution.** (Alternatively, since $\dim C(A) = 2$, by the Rank-Nullity theorem $\dim N(A) = 2 - \dim C(A) = 0$, so $N(A) = \{\mathbf{0}\}$).

3. (a) (4 points) Say what it means for a list of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ to be *linearly independent.*

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is *linearly independent* if the only solution to the equation

$$c_1\mathbf{v}_1 + \ldots + \mathbf{v}_k = \mathbf{0}$$

is all $c$'s equal to 0.

An equivalent definition is as follows: for $k > 1$, a set of vectors is linearly independent if you can't write any one vector as a linear combination of the others. If $k = 1$, a vector is linearly independent if it is not the zero vector.

(b) (6 points) Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in $\mathbb{R}^n$. Is the collection $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-2\mathbf{v}, \mathbf{w}+\mathbf{u}-\mathbf{v}\}$ linearly independent or linearly dependent? Show your work.

We want to solve

$$c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{u}-2\mathbf{v}) + c_3(\mathbf{w}+\mathbf{u}-\mathbf{v}) = \mathbf{0}$$

Rearranging the terms we get :

$$(c_1 + c_2 + c_3)\mathbf{u} + (c_1 - 2c_2 + c_3)\mathbf{v} + (c_3)\mathbf{w} = \mathbf{0}$$

Because $\mathbf{w}, \mathbf{u}, \mathbf{v}$ are linearly independent it follows that the coefficients in the equation above must all be 0. This implies that that

$$c_1 + c_2 + c_3 = 0 \quad (5)$$
$$c_1 - 2c_2 + c_3 = 0. \quad (6)$$
$$c_3 = 0 \quad (7)$$

The last equation implies that $c_3 = 0$; substituting this into the first two equations, and solving by a method of one's choice (for instance, using substitution) implies that $c_1 = c_2 = 0$. Therefore since $c_1 = c_2 = c_3 = 0$ was the only solution, our initial vectors are linearly independent.

4. (a) (4 points) Say what it means for a function $T : \mathbb{R}^n \to \mathbb{R}^m$ to be a *linear transformation*.

The function $T$ as above is a *linear transformation* if the following two conditions are satisfied:

- $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$; and
- $T(c\mathbf{x}) = cT(\mathbf{x})$ for any vector $\mathbf{x} \in \mathbb{R}^n$ and any scalar $c \in \mathbb{R}$.

(b) (5 points) Let $\mathbf{v} \in \mathbb{R}^n$ be any non-zero vector. Recall that on homework, we defined the *projection onto the vector* $\mathbf{v}$, $Proj_{\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}^n$, by the following formula: $\text{Proj}_{\mathbf{v}}(\mathbf{x}) = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$. It is a fact mentioned in class that $Proj_{\mathbf{v}} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a *linear transformation.*

Now, let $P \subset \mathbb{R}^n$ be a plane through the origin, described as $P = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ for a pair of perpendicular non-zero vectors $\mathbf{v}_1, \mathbf{v}_2$. We can define the *projection onto the plane* $P$

$$\text{Proj}_P : \mathbb{R}^n \to \mathbb{R}^n$$

via the following formula:

$$\text{Proj}_P(\mathbf{x}) := \text{Proj}_{\mathbf{v}_1}(\mathbf{x}) + \text{Proj}_{\mathbf{v}_2}(\mathbf{x}).$$

Show directly from definitions that $\text{Proj}_P$ is a linear transformation. You may use the fact that $\text{Proj}_{\mathbf{v}}$ is a linear transformation for any $\mathbf{v}$ without justification (though you should state how you are using it).

We check the conditions in the previous part:

- For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\begin{aligned}
\text{Proj}_P(\mathbf{x}+\mathbf{y}) &= \text{Proj}_{\mathbf{v}_1}(\mathbf{x}+\mathbf{y}) + \text{Proj}_{\mathbf{v}_2}(\mathbf{x}+\mathbf{y}) && (8)\\
&= (\text{Proj}_{\mathbf{v}_1}(\mathbf{x}) + \text{Proj}_{\mathbf{v}_1}(\mathbf{y})) + (\text{Proj}_{\mathbf{v}_2}(\mathbf{x}) + \text{Proj}_{\mathbf{v}_2}(\mathbf{y})) && (9)\\
&= (\text{Proj}_{\mathbf{v}_1}(\mathbf{x}) + \text{Proj}_{\mathbf{v}_2}(\mathbf{x})) + (\text{Proj}_{\mathbf{v}_1}(\mathbf{y}) + \text{Proj}_{\mathbf{v}_2}(\mathbf{y})) && (10)\\
&= \text{Proj}_P(\mathbf{x}) + \text{Proj}_P(\mathbf{y}). && (11)
\end{aligned}$$

where the first line uses the definition of $\text{Proj}_P$, the second line used the fact that $\text{Proj}_{\mathbf{v}_1}$ and $\text{Proj}_{\mathbf{v}_2}$ were linear transformations, the third line redistributed terms, and the last line is again the definition of $\text{Proj}_P$.

- For any vector $\mathbf{x} \in \mathbb{R}^n$ and any scalar $c \in \mathbb{R}$,

$$\begin{aligned}
\text{Proj}_P(c\mathbf{x}) &= \text{Proj}_{\mathbf{v}_1}(c\mathbf{x}) + \text{Proj}_{\mathbf{v}_2}(c\mathbf{x}) && (12)\\
&= c\text{Proj}_{\mathbf{v}_1}(\mathbf{x}) + c\text{Proj}_{\mathbf{v}_2}(\mathbf{x}) && (13)\\
&= c(\text{Proj}_{\mathbf{v}_1}(\mathbf{x}) + \text{Proj}_{\mathbf{v}_2}(\mathbf{x})) && (14)\\
&= c\text{Proj}_P(\mathbf{x}). && (15)
\end{aligned}$$

where once more the first line is the definition of $\text{Proj}_P$, the second line uses the fact that $\text{Proj}_{\mathbf{v}_1}$ and $\text{Proj}_{\mathbf{v}_2}$ were linear transformations, the third line factored out the $c$, and and the last line is the definition of $\text{Proj}_P$.

(c) (6 points) Now let $n = 3$ (so we are working in $\mathbb{R}^3$), and let $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, and $P = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$. Compute the matrix associated to $\text{Proj}_P$ in this case.

You may use the following fact if it is helpful: Both $\mathbf{v}_1$ and $\mathbf{v}_2$ are *unit-length vectors*, meaning $||\mathbf{v}_1|| = ||\mathbf{v}_2|| = 1$. (Note that when $\mathbf{m} \in \mathbb{R}^n$ is unit length, the formulae for projection onto $\mathbf{m}$ simplifies to: $\text{Proj}_{\mathbf{m}}(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{m})\mathbf{m}$).

For any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, the matrix associated to $T$ is

$$\begin{bmatrix} | & | & \cdots & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & \cdots & | \end{bmatrix}.$$

Thus, the matrix for $\text{Proj}_P : \mathbb{R}^3 \to \mathbb{R}^3$ is $\begin{bmatrix} | & | & | \\ \text{Proj}_P(\mathbf{e}_1) & \text{Proj}_P(\mathbf{e}_2) & \text{Proj}_P(\mathbf{e}_3) \\ | & | & | \end{bmatrix}$, so we simply need to compute

Using the simplified formula above, since $\mathbf{v}_1$ and $\mathbf{v}_2$ are both unit-length vectors, we compute that

$$\begin{aligned} \text{Proj}_P(\mathbf{e}_1) &= \text{Proj}_{\mathbf{v}_1}(\mathbf{e}_1) + \text{Proj}_{\mathbf{v}_2}(\mathbf{e}_1) && (16) \\ &= \left(\mathbf{e}_1 \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} + \left(\mathbf{e}_1 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} && (17) \\ &= (1/\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} + (1/\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} && (18) \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{e}_1. && (19) \end{aligned}$$

Similarly,

$$\begin{aligned} \text{Proj}_P(\mathbf{e}_2) &= \text{Proj}_{\mathbf{v}_1}(\mathbf{e}_2) + \text{Proj}_{\mathbf{v}_2}(\mathbf{e}_2) && (20) \\ &= \left(\mathbf{e}_2 \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} + \left(\mathbf{e}_2 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} && (21) \\ &= (-1/\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} + (1/\sqrt{2}) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} && (22) \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{e}_2. && (23) \end{aligned}$$

Finally,

$$\begin{aligned} \text{Proj}_P(\mathbf{e}_3) &= \text{Proj}_{\mathbf{v}_1}(\mathbf{e}_3) + \text{Proj}_{\mathbf{v}_2}(\mathbf{e}_3) && (24) \\ &= \left(\mathbf{e}_3 \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} + \left(\mathbf{e}_3 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} && (25) \\ &= (0) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} + (0) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} && (26) \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}, && (27) \end{aligned}$$

So, the matrix associated to $\text{Proj}_P$ is

$$\begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{0} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

5. (12 points) Find, showing all your work and using any method of your choice, all solutions (in vector form) to the system of equations:

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & 1 & -8 \\ 5 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ -3 \end{bmatrix}.$$

Describe your solutions as a point, line, or plane.

We solve the system by writing the associated augmented matrix: The augmented matrix is :

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ -2 & 1 & -8 & -4 \\ 5 & 4 & 7 & -3 \end{array}\right] \xrightarrow[R_2+2R_1]{R_3-3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 5 & -10 & -10 \\ 0 & -6 & 12 & 12 \end{array}\right] \xrightarrow[R2/5]{-R_3/6} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & -2 & -2 \end{array}\right] \xrightarrow[R_3-R_2]{R_1-2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have two pivots variables $x_1$ and $x_2$; solving for them in terms of the free variable $x_3$ (using the rref system), we see that $x_1 = -3x_3 + 1$ and $x_2 = 2x_3 - 2$. Thus, letting the free variable $x_3$ range over $\mathbb{R}$, the set of solutions is

$$\left\{ \begin{bmatrix} -3x_3 + 1 \\ 2x_3 - 2 \\ x_3 \end{bmatrix}, x_3 \in \mathbb{R} \right\} = \left\{ x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, x_3 \in \mathbb{R} \right\}$$

This is the parametric representation of a line.

6. (12 points) (2 points each) Each of the statements below is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). For each part, decide which and circle the appropriate choice; you *do not* need to justify your answers.

(a) Let $A$ be a $3 \times 4$ matrix. If $\dim C(rref(A)) = 2$, then $\dim C(A) = 2$. **[T]** F MAYBE

**True**. The dimension of $C(A)$ is the number of pivots in $rref(A)$, which is also the dimension of $C(rref(A))$.

(b) If $P \subset \mathbb{R}^n$ is a plane going through the origin, and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $P$, then $P$ is spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. T F **[MAYBE]**

**Maybe**. If at least two of them are linearly independent, then yes. If all of them are collinear, then no.

(c) If the reduced row echelon form of the augmented matrix $[A|\mathbf{b}]$ has two free variables, then there are many solutions to the associated system. T F **[MAYBE]**

**Maybe**. If $\mathbf{b}$ is in the column space of $A$, then yes. Otherwise, there are no solutions.

(d) If $A$ is a $2 \times 3$ matrix, then $N(A) \neq \{\mathbf{0}\}$. **[T]** F MAYBE

**True**. There are at most two pivots for $A$, so there has to be at least one free variable, which implies the dimension of the null space is at least 1. Therefore there are infinitely many elements in $N(A)$, so indeed $N(A) \neq \{\mathbf{0}\}$.

(e) If $A$ is a $3 \times 2$ matrix, then $N(A) \neq \{\mathbf{0}\}$. T F **[MAYBE]**

**Maybe**. If $rref(A)$ has two pivots (equivalently if the two columns of $A$ are linearly independent), then $N(A) = \{\mathbf{0}\}$. If $rref(A)$ has one or fewer pivor variables (equivalently if the two columns of $A$ are linearly dependent), then indeed $N(A) \neq \{\mathbf{0}\}$.

(f) Suppose $V$ is a subspace in $\mathbb{R}^n$ of dimension $d$, and a collection of $d$ vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ spans $V$. Then this collection is also linearly independent. **[T]** F MAYBE

**True**. This is an immediate application of Levandosky Proposition 12.3.