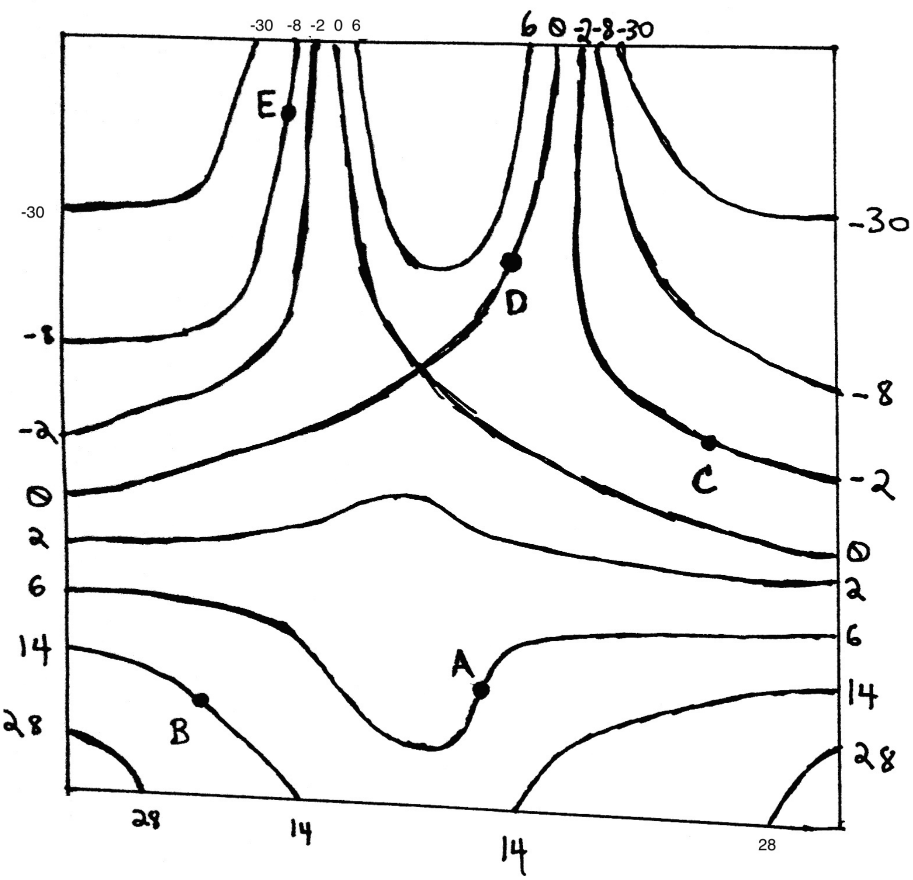
Solutions to Math 51 Midterm 2 — July 27, 2016

1. (12 points) (2 points each) The diagram below shows several marked points on the contour map of a function $f(x, y)$. You may assume that f has continuous first and second derivatives, and that the scales on the x and y axes, which are parallel to the edges of the box, are the same. The numbers drawn are the heights of the various level sets.



Circle the appropriate word to complete each sentence (there is a unique *best answer* based on the diagram in each case). No justification is necessary.

(a) At the point A the value of $\frac{\partial f}{\partial x}$ is POSITIVE ZERO NEGATIVE.

POSITIVE

(b) At the point A, the value of $\frac{\partial f}{\partial y}$ is POSITIVE ZERO NEGATIVE.

NEGATIVE

(c) At the point B, the value of $\frac{\partial f}{\partial x}$ is POSITIVE ZERO NEGATIVE.

NEGATIVE

(d) At the point C, the value of $\frac{\partial^2 f}{\partial y^2}$ is POSITIVE NEGATIVE.

NEGATIVE

(e) At the point D, the value of $\frac{\partial^2 f}{\partial x \partial y}$ is POSITIVE NEGATIVE.

NEGATIVE

(f) At the point E, the value of $\frac{\partial^2 f}{\partial x^2}$ is POSITIVE NEGATIVE.

NEGATIVE

2. (a) (6 points) Let $\mathbf{r}(t) = \begin{bmatrix} \frac{1}{1+t^2} \\ \frac{1}{e^{t-1}} \end{bmatrix}$. Find a parametric representation of the tangent line to the image of $\mathbf{r}$ at $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

If $\mathbf{r}(t) = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, then $t = 1$. We find the derivative of $\mathbf{r}(t)$ using the chain rule:

$$\mathbf{r}'(t) = \begin{bmatrix} \frac{-2t}{(1+t^2)^2} \\ -\frac{1}{e^{t-1}} \end{bmatrix} \Longrightarrow \mathbf{r}'(1) = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix}$$

It follows that the parametric representation of the tangent line at $t = 1$ is

$$L = \left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix}, t \in \mathbb{R} \right\}$$

(b) (6 points) Find, using a method of your choice, an equation (of the form $ax + by + cz = d$) for the tangent plane to the surface $\{(x, y, z) | z = 2 + x^3 - 2xy^2\}$ at the point $(1, 1, 1)$.

We can write the equation above as $2 + x^3 - 2xy^2 - z = 0$. Set $F(x, y, z) = 2 + x^3 - 2xy^2 - z$. We are finding the tangent plane to the level set of F of height 0 at the point $(1, 1, 1)$. First we find the gradient of F:

$$\nabla F(x, y, z) = \begin{bmatrix} 3x^2 - 2y^2 \\ -4xy \\ -1 \end{bmatrix} \Longrightarrow \nabla F(1, 1, 1) = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

The equation of the plane will be $\nabla F(1, 1, 1) \cdot (\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = 0$ which is equivalent to:

$$(x - 1) - 4(y - 1) - (z - 1) = 0 \iff x - 4y - z = -4$$

3. Let $Ref_L : \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation which is reflection across the line $L = \text{span}(\left[\begin{smallmatrix} 1 \\ -3 \end{smallmatrix}\right])$.

(a) (6 points) Let $\mathcal{B} = \{\left[\begin{smallmatrix} 1 \\ -3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]\}$. Find the matrix B for Ref_L in $\mathcal{B}$-coordinates (in class, we called this matrix $[\mathcal{M}(Ref_L)_{\mathcal{B}}]$).

Since Ref_L is reflection across the span of the vector $\mathbf{w}_1 = \left[\begin{smallmatrix} 1 \\ -3 \end{smallmatrix}\right]$, when it is applied to $\mathbf{w}_1$, $\mathbf{w}_1$ remains unchanged. $\mathbf{w}_2 = \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$ is orthogonal to $\mathbf{w}_1$, so if we reflect it we get $-\mathbf{w}_1$. This means $Ref_L(\mathbf{w}_1) = 1 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2$ and $Ref_L(\mathbf{w}_2) = 0 \cdot \mathbf{w}_1 + (-1) \cdot \mathbf{w}_2$. In $\mathcal{B}$-coordinates, the matrix for our transformation will look like $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) (6 points) Find, using any method of your choice (but with justification), the matrix A for the linear transformation Ref_L (in standard coordinates).

Let $C = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$. Then $A = CBC^{-1}$. We need to find C^{-1}. Since C is 2×2, $C = \frac{1}{\det(C)} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$. We compute

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\det(C)} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

(c) (4 points) What is the matrix for the composition $Ref_L \circ Ref_L$?

Notice that B^2 is the identity. That means that the matrix for $Ref_L \circ Ref_L$ will be the identity in any basis.

4. (16 points) Find, with justification, an *orthonormal eigenbasis* for the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$.

First we find the eigevalues of A by solving $\det(\lambda I - A) = 0$. We have

$$\det\left(\begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 1 & -3 \\ 0 & -3 & \lambda - 1 \end{bmatrix}\right) = (\lambda - 1)\det\left(\begin{bmatrix} \lambda - 1 & -3 \\ -3 & \lambda - 1 \end{bmatrix}\right) =$$

$$= (\lambda - 1)[(\lambda - 1)^2 - 9] = (\lambda - 1)(\lambda - 4)(\lambda + 2)$$

Therefore the eigenvalues are $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = -2$. We find the corresponding eigenspaces:

E_{λ_1}: $\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & 1-1 & -3 \\ 0 & -3 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

It follows $E_{\lambda_1} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$. Pick $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

E_{λ_2}: $\begin{bmatrix} 4-1 & 0 & 0 \\ 0 & 4-1 & -3 \\ 0 & -3 & 4-1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

It follows $E_{\lambda_2} = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$. Pick $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The factor in front is so that we have a unit vector.

E_{λ_3}: $\begin{bmatrix} -2-1 & 0 & 0 \\ 0 & -2-1 & -3 \\ 0 & -3 & -2-1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

It follows $E_{\lambda_3} = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$. Pick $\mathbf{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

We already know the three vectors we chose are orthogonal since they are eigenvectors of a symmetric matrix corresponding to different eigenvalues. Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \}$ form an orthonormal eigenbasis.

5. Suppose the temperature at a point in space is described by the following function of $\mathbb{R}^3$:

$$T(x, y, z) = x^3y^2 - \cos(x^2z)$$

(a) (6 points) Imagine a bee is flying in space and is at the point $(1, 2, \frac{\pi}{2})$. If the bee is flying in the unit direction $\frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\\0\end{bmatrix}$, is the temperature increasing, decreasing, or staying the same?

We need to find the directional derivative of T at $\mathbf{a} = (1, 2, \frac{\pi}{2})$ in the direction $\mathbf{w} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\\0\end{bmatrix}$. For this we find the gradient

$$\nabla T(x, y, z) = \begin{bmatrix}3x^2y^2 + 2xz\sin(x^2z)\\2x^3y\\x^2\sin(x^2z)\end{bmatrix} \Longrightarrow \nabla T(1, 2, \frac{\pi}{2}) = \begin{bmatrix}12+\pi\\4\\1\end{bmatrix}$$

It follows that $DT_{\mathbf{w}}(\mathbf{a}) = \begin{bmatrix}12+\pi\\4\\1\end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\\0\end{bmatrix} = \frac{1}{\sqrt{5}}(4+\pi) > 0$. Therefore the temperature is increasing in the direction $\mathbf{w}$.

(b) (4 points) The bee is feeling too hot at $(1, 2, \frac{\pi}{2})$. What (not necessarily unit) direction should the bee fly in to decrease the temperature as quickly as possible?

The bee should move in the direction $-\nabla T(1, 2, \frac{\pi}{2}) = \begin{bmatrix}-12-\pi\\-4\\-1\end{bmatrix}$.

(c) (5 points) Suppose the *humidity* at a point in space is described by the function

$$H(x,y,z) = x^3 - y^2 + z^2.$$

Set up (but do not solve) for a system of equations (or inequalities) for a unit direction $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ which the bee could fly in from the point $(1, 2, \frac{\pi}{2})$ in which the humidity would stay the same and temperature would increase.

First we compute the gradient of H:

$$\nabla H(x,y,z) = \begin{bmatrix} 3x^2 \\ -2y \\ 2z \end{bmatrix} \Longrightarrow \nabla H(1,2,\frac{\pi}{2}) = \begin{bmatrix} 3 \\ -4 \\ \pi \end{bmatrix}$$

In order for the humidity to stay constant and temperature to increase in the direction $\mathbf{v}$, we need $\nabla H(1,2,\frac{\pi}{2}) \cdot \mathbf{v} = 0$ and $\nabla T(1,2,\frac{\pi}{2}) \cdot \mathbf{v} > 0$. We can expand the dot product and write this as the following (in)equalities:

$$3v_1 - 4v_2 + \pi v_3 = 0$$

$$(12 + \pi)v_1 + 4v_2 + v_3 > 0$$

6. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the following function:

$$f(x, y) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^x \\ e^y + y \end{bmatrix}. \tag{1}$$

You may assume without justification that f is differentiable.

(a) (4 points) Compute the matrix of partial deriatives of f at $(0, 0)$, $Df(0, 0)$.

We have $f(x, y) = \begin{bmatrix} e^x + e^y + y \\ 2e^y + 2y \end{bmatrix}$. Therefore, if we let f_1 denote the first component of f and f_2 the second component, then

$$Df(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} e^x & e^y + 1 \\ 0 & 2e^y + 2 \end{bmatrix}.$$

So

$$Df(0, 0) = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}.$$

(b) (6 points) Using the fact that f has a linearization, estimate $f(-0.1, 0.1)$.

The linearization $L(x, y)$ of f at $(0, 0)$ is given by

$$L(x, y) = f(0, 0) + Df(0, 0)\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 + x + 2y \\ 2 + 4y \end{bmatrix}.$$

Therefore,

$$f(-0.1, 0.1) \approx L(-0.1, 0.1) = \begin{bmatrix} 2 + (-0.1) + 2(0.1) \\ 2 + 4(0.1) \end{bmatrix} = \begin{bmatrix} 2.1 \\ 2.4 \end{bmatrix}.$$

(c) (6 points) Let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be a differentiable function, such that $g(1, 1, 3) = (0, 0)$. Suppose that $Dg(1, 1, 3) = \left[\begin{smallmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{smallmatrix}\right]$, calculate $D(f \circ g)(1, 1, 3)$.

By the chain rule,

$$D(f \circ g)(1, 1, 3) = Df(g(1, 1, 3))Dg(1, 1, 3) = Df(0, 0)\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}.$$

We use the computation of $Df(0, 0)$ in part (a) to obtain

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 7 \\ 8 & -4 & 12 \end{bmatrix}$$

So

$$D(f \circ g)(1, 1, 3) = \begin{bmatrix} 5 & -2 & 7 \\ 8 & -4 & 12 \end{bmatrix}$$

7. (12 points) (2 points each) Each of the statements below is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). For each part, decide which and circle the appropriate choice; you *do not* need to justify your answers.

(a) If the partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}$ all exist at $\mathbf{a} \in \mathbb{R}^n$, then f is differentiable at $\mathbf{a}$. T F **[MAYBE]**

Not necessarily true. See problem 58 from chapter 2.3. (Note that this was a homework problem).

(b) If an $n \times n$ matrix A is diagonalizable, then so is A^3. **[T]** F MAYBE

If A is diagonalizable, then it has an eigenbasis. The same vectors will also form an eigenbasis for A^3, though the corresponding eigenvalues might be different.

(c) If an $n \times n$ matrix A^2 is diagonalizble, so is A. T F **[MAYBE]**

Let $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. One can check A_1 does not have an eigenbasis while A_2 is already diagonal. If we square both of them, we get A_2 which is diagonalizable.

(d) There are functions $f : \mathbb{R}^2 \to \mathbb{R}$ which are not continuous at at $(0,0)$ but are still differentiable at $(0,0)$. T **[F]** MAYBE

False by the theorem 3.6 in chapter 2.3 in the book.

(e) If a 3×3 matrix A only has -1 and 2 as eigenvalues, then A is invertible. **[T]** F MAYBE

A is invertible if and only if it does NOT have 0 as an eigenvalue. Since A only has -1 and 2 as eigenvalues, it follows it must be invertible.

(f) If a 3×3 matrix A has only -1 and 2 as eigenvalues, then A is diagonalizable. T F **[MAYBE]**

Not necessarily. For example one can check the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is not diagonalizable, because the eigenspaces corresponding to $\lambda = 2$ and $\lambda = -1$ are only one dimensional. Thus, it is not possible to find an eigenbasis for A, meaning A is not diagonalizable.