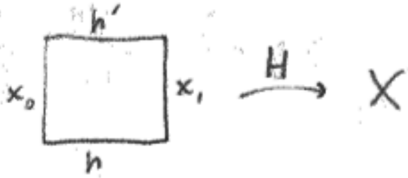
MATH 215B Winter 2013

1. We could take

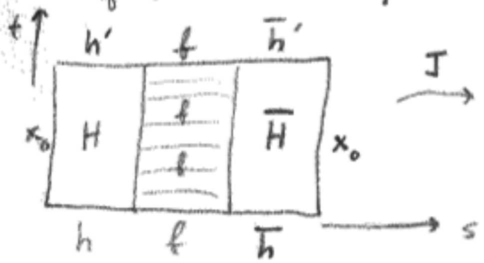
$$(\mathbb{R}^n - \{0\}) \times I \longrightarrow \mathbb{R}^n - \{0\}$$

$$(x,t) \longmapsto (1-t)x + t\frac{x}{\|x\|}$$

2. If h is homotopic to h' via



then $h \cdot f \cdot \bar{h}$ is homotopic to $h' \cdot f \cdot \bar{h}'$ by



or in coordinates

$$J(s,t) = \begin{cases} H(3s,t) & s \le \frac{1}{3} \\ f(3s-1) & \frac{1}{3} \le s \le \frac{2}{3} \\ H(3-3s,t) & \frac{2}{3} \le s \end{cases}$$

So if $h \sim h'$ then $\beta_h = \beta_{h'}$.

3. The quotient $(S^1 \times I)/(S^1 \times \{1\})$ is homeomorphic to D^2 via $(x,t) \mapsto (1-t)x$. (Easy to check it's a continuous bijection, and both spaces are compact Hausdorff so we're done.) This lets us turn a nullhomotopy of $S^1 \to X$ into an extension to $D^2 \to X$ and vice-versa, so (a) $\Leftrightarrow$ (b). Then (c) $\Rightarrow$ (a) is obvious, and (b) $\Rightarrow$ (c) is proved using

$$S^1 \times I \xrightarrow{f} D^2$$

$$x, t \longmapsto (1-t)x + t(1,0)$$

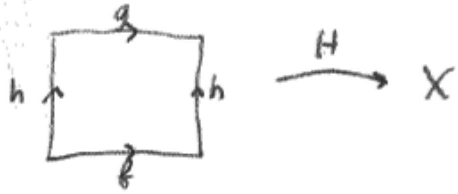
Note that $\{(1,0)\} \times I$ and $S^1 \times \{1\}$ both go to the basepoint, so composing a map $D^2 \to X$ with f gives a based homotopy of the map $S^1 \hookrightarrow D^2 \to X$ to the constant map, proving (b) $\Rightarrow$ (c).

If all maps $S^1 \to X$ are homotopic, then $\forall x, y \in X$, $S^1 \to \{x\} \to X$ is homotopic to $S^1 \to \{y\} \to X$ via $H: S^1 \times I \to X$. Then $H|_{* \times I}$ is a path from x to y, so X is path-connected. From (a) $\Rightarrow$ (c), X is simply connected.

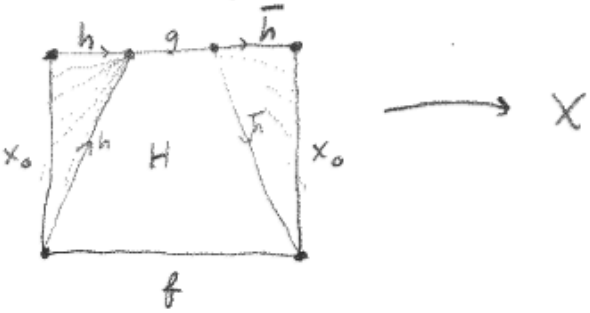
If X is simply connected, then all loops are homotopic to constant loops since every $\pi_1(X,x) = 0$. All constant loops are homotopic: if γ is a path from x to y then $H(x,t) = \gamma(t)$ is a homotopy from $S^1 \to \{x\} \to X$ to $S^1 \to \{y\} \to X$. Therefore all loops are homotopic.

4. Let X be path connected. If $S^1 \xrightarrow{f} X$ is a loop based at x_1, then there exists a path h from x_0 to x_1. From the section f is homotopic to $h \cdot f \cdot \bar{h}$, so $[f] = \Phi([h \cdot f \cdot \bar{h}])$ and Φ is onto.

Now let f, g be loops based at x_0. If $\Phi([f]) = \Phi([g])$ then there is a homotopy



Here $h(t) = H(*, t)$ is the path traced out by the basepoint of S^1. It is also a closed loop at x_0. Reparametrize H to get



This shows $[f] = [h][g][h]^{-1}$ in $\pi_1(X, x_0)$.

5. Lemma 1.19 gives

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "id"] \arrow[dr, "id"'] & \pi_1(X,x_0) \arrow[d, "\beta_h"] \\ & \pi_1(X,x_0) \end{tikzcd}$$

so $\beta_h = id$, where $h(t) = f_t(x_0)$.

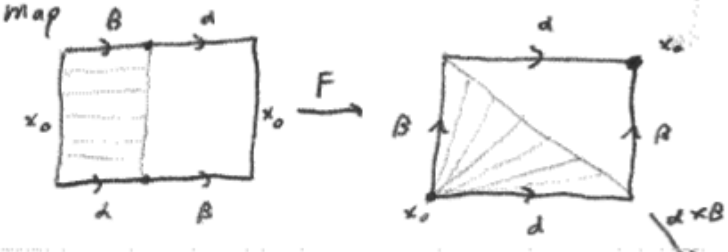
So for any $\alpha \in \pi_1(X, x_0)$,

$[\alpha] = [h][\alpha][h^{-1}]$

$[\alpha][h] = [h][\alpha]$

So $h \in Z(\pi_1(X, x_0))$.

6. If $\alpha : S^1 \to X$ and $\beta : S^1 \to Y$ are our two loops then $\alpha \times \beta : S^1 \times S^1 \to X \times Y$ puts a torus into $X \times Y$. Intuitively, it is the 2-cell of this torus that gives our homotopy. More precisely, form the map



$$I^2 \xrightarrow{F} I^2 \xrightarrow{\alpha \times \beta} X \times Y$$

$$F(s,t) = \begin{cases} (2s(1-t), 2st) & s \le 1/2 \\ (1 + (2s-2)t, 1 + (2s-2)(1-t)) & s \ge 1/2 \end{cases}$$

This is an explicit homotopy. :-)

7. (a) $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(\mathbb{R}^3) \cong 0$

If $S^1 \cong A \hookrightarrow \mathbb{R}^3$ has a retract then $\mathbb{Z} \hookrightarrow 0$ ✗

(b) If $S^1 \times S^1 \hookrightarrow S^1 \times D^2$ has a retract then

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[id \oplus 0]{} \mathbb{Z} \oplus 0$$

is injective, ✗

8. $p: \tilde{X} \to X$ finite cover.

If $\tilde{X}$ is compact Hausdorff then X is the cts. image of a compact space, therefore compact. If $x \neq y$, $x, y \in X$, take evenly covered $U \ni x$ and $V \ni y$. For each preimage x_i of x and y_i of y, take $U_{ij} \ni x_i$ and $V_{ij} \ni y_j$, $U_{ij} \cap V_{ij} = \phi$. Then $\bigcap_i p(p^{-1}(U) \cap \bigcap_j U_{ij})$ and $\bigcap_j p(p^{-1}(V) \cap \bigcap_i V_{ij})$ are two disjoint open sets about x and y, so X is Hausdorff.

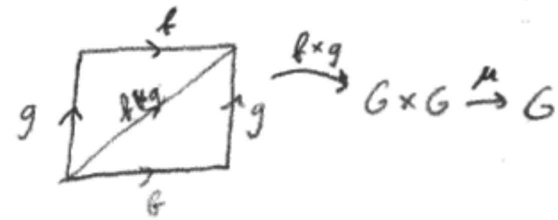
If X is compact Hausdorff and $x, y \in \tilde{X}$, $x \neq y$, then either $p(x) \neq p(y)$, so take disjoint $U \ni p(x)$ and $V \ni p(y)$ and take $p^{-1}(U)$, $p^{-1}(V) \subseteq \hat{X}$, or $p(x) = p(y)$, in which case take two distinct homeomorphic copies of $U \ni p(x)$ in $\hat{X}$. This shows $\tilde{X}$ is Hausdorff.

Since X is locally compact, every open $U \subseteq X$ is covered by open $\{V_i\}$ such that $\overline{V_i} \subseteq U$ $\forall i$. Doing this to the evenly covered sets, we get a cover of X by open sets V_i such that $\overline{V_i}$ is compact and evenly covered. WLOG there are finitely many V_i. Each $\overline{V_i}$ has finitely many homeomorphic preimages. This covers $\hat{X}$ by finitely many compact sets, so $\tilde{X}$ is compact.

9. (a) $f: S^1 \to G$

$g: S^1 \to G$

$f \times g: S^1 \times S^1 \to G \times G$



It is easy to check that $f \cdot g$ is obtained by following the bottom right path of the square, $g \cdot f$ by following the top left, and $f * g$ by following the diagonal. (After applying $\mu \circ (f \times g)$.) Take a straight-line homotopy in I^2 between any of these paths, and apply $\mu \circ (f \times g)$. This shows that $f \cdot g$, $f * g$, and $g \cdot f$ are homotopic.