

# Math 215B Homework 2

Due January 25th, 2013 by 5 pm

Please remember to write down your name and Stanford ID number (9 digits). All pages and sections refer to pages and sections in Hatcher's *Algebraic Topology*.

1. (14 points) *A version of Van Kampen's theorem for computing  $\pi_1(S^1)$ .* One shortcoming of the Van Kampen theorem as discussed in class is the requirement that the intersections  $U_\alpha \cap U_\beta$  be connected in order to apply the theorem to the cover  $\{U_\alpha, U_\beta\}$ . This means that we cannot, e.g., apply Van Kampen to the decomposition of  $S^1$  into the union of two open intervals e.g.  $U_1 = \exp(2\pi i(-.1, .6))$ ,  $U_2 = \exp(2\pi i(.5, 1))$ . In this exercise, we will develop a version of Van Kampen's theorem that will allow us to compute  $\pi_1(S^1)$  using the decomposition  $S^1 = U_1 \cup U_2$ . First, some terminology:

**DEFINITION 0.1.** *Let  $S$  be a (finite) set. A **groupoid** over  $S$  is a set  $G$  of elements indexed by pairs of elements of  $S$ ,  $G = \bigcup_{i,j \in S} G_{ij}$ , with "multiplication" maps*

$$(0.1) \quad \cdot : G_{ij} \times G_{jk} \rightarrow G_{ik}$$

*satisfying the following properties:*

- *(identity) For each  $i$ , there exists an element  $e_i \in G_{ii}$  such that for any  $\gamma \in G_{ij}$ ,  $e_i \cdot \gamma = \gamma$  and  $\gamma \cdot e_j = \gamma$ .*
- *(associativity) Let  $\alpha \in G_{ij}$ ,  $\beta \in G_{jk}$  and  $\gamma \in G_{kl}$ . The multiplication of these three elements is associative  $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$ .*
- *(inverses) Let  $\alpha \in G_{ij}$ . Then, there exists an  $\bar{\alpha} \in G_{ji}$  such that  $\alpha \cdot \bar{\alpha} = e_i$ , and  $\bar{\alpha} \cdot \alpha = e_j$ .*

*Note that  $\cdot$  gives each subset  $G_{ii}$  the structure of a group with identity element  $e_i$ . A pair of elements  $\alpha, \beta \in G$  are said to be **composable** if  $\alpha \in G_{ij}$  and  $\beta \in G_{jk}$  for some  $i, j, k$  (so that the multiplication  $\alpha \cdot \beta$  is defined). A **homomorphism** between groupoids is a collection of maps*

$$(0.2) \quad \phi : G_{ij} \rightarrow H_{ij} \text{ for each } i, j$$

*compatible with multiplications as follows: if  $\alpha, \beta$  are composable, then*

$$(0.3) \quad \phi(\alpha \cdot \beta) = \phi(\alpha) \cdot \phi(\beta).$$

*An **isomorphism** of groupoids over  $S$  is a homomorphism such that each map (0.2) is a bijection.*

The application to us is in the following enlargement of the fundamental group. Let  $X$  be a topological space and let  $\mathcal{P}_{xy}$  denote the set of homotopy classes of paths between points  $x$  and  $y$  (so  $\mathcal{P}_{xx} = \pi_1(X, x)$ ). Now let  $A = \{x_1, \dots, x_k\}$  be a collection of points in  $X$ . Define the **fundamental groupoid**  $\pi_1(X, A)$  to be the following groupoid over  $A$ :

$$(0.4) \quad \pi_1(X, A)_{ij} := \mathcal{P}_{x_i x_j},$$

with multiplication of composable elements given by composition of paths.

- (a) (2 points) Verify that  $\pi_1(X, A)$  is indeed a groupoid over  $A$ . You may use any results we have proved in class.

- (b) (3 points) *Free products of groupoids*: Let  $G_1$  and  $G_2$  be two groupoids over a finite set  $S$ . Define the *free product* groupoid  $G_1 \star^S G_2$  as follows:  $(G_1 \star^S G_2)_{ij}$  is the equivalence class of words of the form

$$(0.5) \quad w_1 \cdots w_k | w_i \in (G_{n_i})_{m_{i-1}, m_i} \text{ for some } n_i \in \{1, 2\} \text{ and } m_j \in S, m_0 = i, m_k = j$$

such that that no letter is an identity  $e_i$ , modulo the equivalence of multiplication of adjacent letters which are composable and in the same groupoid (and removal of identity elements if they appear). Prove that the result satisfies the axioms of a groupoid (Hint for associativity: define a reduced representative of a word and prove that it is unique).

- (c) (5 points) Now, suppose a path-connected space  $X$  decomposes as  $U_1 \cup U_2$  where the intersection  $U_1 \cap U_2$  is a disconnected union of two simply-connected components,  $U_1 \cap U_2 = W \amalg V$ . Choose a pair of points  $A = \{x_1, x_2\}$  with  $x_1 \in W$ , and  $x_2 \in V$ . Adapting the proof of Van Kampen's theorem from class (or Hatcher, see pages 43-45), prove that

$$(0.6) \quad \pi_1(X, A) \cong \pi_1(U_1, A) \star^A \pi_1(U_2, A).$$

This is a special case of a general Van Kampen theorem for fundamental groupoids.

- (d) (4 points) Finally, let  $X = S^1$ , and  $U_1, U_2$  the two open sets considered in the preamble. Let  $p$  and  $q$  be a point in each connected component of  $U_1 \cap U_2$ , and  $A = \{p, q\}$ . Prove, using the above strengthening of Van Kampen's theorem, that  $\pi_1(S^1, p) \cong \mathbb{Z}$ . (Note:  $\pi_1(S^1, p)$  by definition is  $\pi_1(S^1, A)_{11}$ , using the notation above).
2. (6 points) *Constructing CW complexes*. Solve page 21 (§0), problem 20.
3. (6 points) *Topology of CW complexes*. Let  $X$  be a CW complex with one 0-cell and infinitely many 1-cells. Show that the topology of  $X$  is not metrizable (we cannot make  $X$  into a metric space in such a way that its metric topology is its CW topology). More generally, show that the same applies to any CW complex  $X$  in which some point belongs to the closure of infinitely many cells.
4. (6 points) *Attaching higher dimensional cells does not affect  $\pi_1$* . In class we asserted that adding cells of dimension  $\geq 3$  to a CW complex does not affect the fundamental group. Prove this assertion (namely, solve page 53 (§1.2), problem 6).
5. (6 points) Solve page 53 (§1.2), problem 8.
6. (6 points) Solve page 53 (§1.2), problem 11.
7. (6 points) Solve page 53 (§1.2), problem 12.
8. (5 points) *Homotopy extension property*. Let  $X$  be a space and  $A \subset X$  a subspace. The pair  $(X, A)$  is said to satisfy the **homotopy extension property** if, given a homotopy of maps  $F : A \times I \rightarrow Z$  with an extension  $\tilde{f}_0 : X \rightarrow Z$  of  $F|_{A \times \{0\}}$ , there is a common extension  $\tilde{F} : X \times I \rightarrow Z$  of  $F$  and  $\tilde{f}_0$  (i.e.  $\tilde{F}|_{A \times I} = F$  and  $\tilde{F}|_{X \times \{0\}} = \tilde{f}_0$ ). This property is a special case of the notion of a **cofibration**, a notion which may arise later this quarter. Many pairs of spaces  $(X, A)$  satisfy the homotopy extension property, for example take  $X$  a CW complex and  $A$  any subcomplex (For a proof, read §0 Proposition 0.16).

Read Hatcher's section on the homotopy extension property (§0 starting on page 14, up to Proposition 0.18). Then solve page 20 (§0), problem 26.

9. (5 points) A space  $A$  is said to be **contractible** if it is homotopy equivalent to a point. It is a basic fact that if a pair  $(X, A)$  satisfies the homotopy extension property and  $A$  is contractible, then the projection map  $X \rightarrow X/A$  is a homotopy equivalence. This is

proved for example in Proposition 0.17 (page 15), but in this exercise you will given an alternate proof. Namely, solve page 20 (§0), problem 27. You will need to understand the **mapping cylinder**  $M_f$  construction of a map  $f : X \rightarrow Y$ , defined on page 2.