MATH 215B HW 2

1.(a) Using the same homotopies as for $\pi_1(X, x_0)$, concatenation of homotopy classes of paths is well-defined, associative, has identity the constant path, and inverse given by reversing the path.

(b) This can be tedious :-) but it is essentially a restatement of the same proof for free products of groups, or Hatcher p. 41-42

(c) The homomorphism

$$\pi_1(U_1, A) *^A \pi_1(U_2, A) \to \pi_1(X, A)$$

is clear. It's surjective because each path $x_i \to x_j$ can be broken into finitely many segments lying entirely in U_1 or V_1. Up to homotopy, then, the path is a composite of elements from the left-hand side, so our map is surjective. For injectivity we apply a similar argument to a homotopy of paths (as in Hatcher p. 45-46)

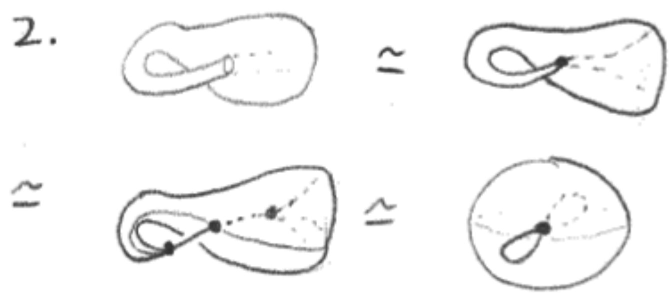
(d) 

Since $U_1 \simeq U_2 \simeq *$ we know all sets in $\pi_1(U_i, A)$ are singletons.

Let a denote the htpy class of $p \to q$ thru U_1, and b the htpy class of $p \to q$ thru U_2. Then $\pi_1(U_1) * \pi_1(U_2)$ consists of reduced words in a and b only. To start and end at p, the word must start at a or b and end at a^{-1} or b^{-1}. This leaves only

$$\begin{array}{lcl} \vdots & & \\ ba^{-1}ba^{-1} & \longmapsto & -2 \\ ba^{-1} & \longmapsto & -1 \\ id & \longmapsto & 0 \\ ab^{-1} & \longmapsto & 1 \\ ab^{-1}ab^{-1} & \longmapsto & 2 \\ \vdots & & \end{array}$$

which is isomorphic to $\mathbb{Z}$.

2. 

$= S^1 \vee S^1 \vee S^2$

3. If $X = \bigvee^{\infty} S^1$ has a metric, construct a nbhd. of the basepoint which on the n^{th} copy of S^1 is a ball of radius $1/n$ about the basepoint. Do this for a countable sequence of S^1's, and for the rest, take the entire S^1. Then this nbhd. is open in the CW topology but not in the metric topology ⨳

For the more general case we again construct a subset open in the CW topology but not the metric topology. Here we must be more careful. The key step is in extending an open subset of S^{n-1} to D^n, as in this picture. We can pick ε each time. Letting $\varepsilon \to 0$ as we extend to more and more cells gives us the desired ~~X_ε~~.

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4. Suppose Y is X with just one extra cell. Apply van Kampen to

$$\begin{tikzcd} (\pi_1=0)\ D^n \arrow[r, hook] & Y \\ (\pi_1=0)\ S^{n-1}\times I \arrow[u, hook] \arrow[r, hook] & \tilde{X} \arrow[u, hook] \end{tikzcd}$$

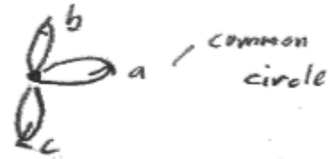
Here $\tilde{X} \simeq X$ is X with an $S^{n-1}\times I$ attached along $S^{n-1} \to X$. Van Kampen tells us $\pi_1(\tilde{X}) \xrightarrow{\sim} \pi_1(Y)$, so $\pi_1(X) \xrightarrow{\sim} \pi_1(Y)$.

Inductively, this is also true for finitely many attached cells. For infinitely many cells, the map is surjective because paths are compact and so land in a finite relative subcomplex $X \hookrightarrow Y'$ (see Hatcher A.1) but $\pi_1(X) \to \pi_1(Y')$ is already surjective. For injectivity, apply this same argument to homotopies of paths.

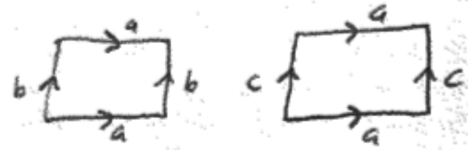
If $X = \mathbb{R}^n -$ a discrete set, $n \geq 3$, then

$X \sim \mathbb{R}^n -$ a disjoint union of open n-cells. Attaching these cells does not change π_1, so $\pi_1 X = 0$.

5. 1-skeleton:



2-cells:



so $\pi_1 X \cong \langle a,b,c \mid aba^{-1}b^{-1}, aca^{-1}c^{-1} \rangle$

$$\cong \underset{a}{\mathbb{Z}} \oplus (\underset{b}{\mathbb{Z}} * \underset{c}{\mathbb{Z}})$$

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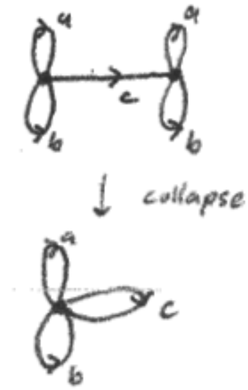
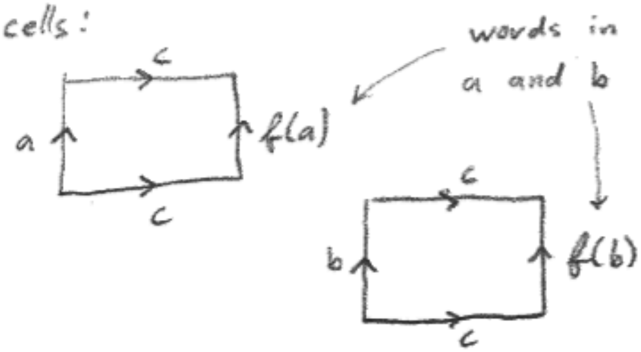
6. 1-skeleton:



2-cells:



so $\pi_1 X \cong \langle a,b,c \mid c f(a) c^{-1} a^{-1},$

$c f(b) c^{-1} b^{-1} \rangle$

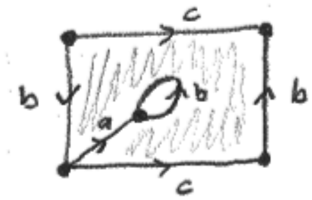
If we start with $S^1 \times S^1$ instead we get an extra 2-cell and an irrelevant 3-cell, giving

$\langle a,b,c \mid c f(a) c^{-1} a^{-1}, c f(b) c^{-1} b^{-1}, aba^{-1}b^{-1} \rangle$

7. From #2, $X \simeq S^1 \vee S^1 \vee S^2$

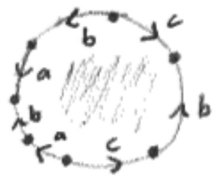
so $\pi_1 X \cong \mathbb{Z} * \mathbb{Z}$

Y has CW structure given by



(all vertices identified)

with a 2-cell in the shaded region. Inspecting this picture, the attaching map is



so $\pi_1 Y \cong \langle a,b,c \mid aba^{-1}b^{-1}cb^{-1}c^{-1} \rangle$

Finally, $\mathbb{R}^3 - Z$ deformation retracts onto Y so $\pi_1 Y \cong \pi_1(\mathbb{R}^3 - Z)$.

8. Let (X,A) have HEP.

Pick a homeomorphism

$$I^2 \xrightarrow{\cong} I^2$$

Multiply by X:

$$X \times I^2 \xrightarrow{\cong} X \times I^2$$

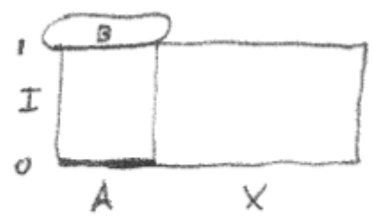
$$\begin{pmatrix} X \times I \times \{0\} \\ \cup\ X \times \{0\} \times I \\ \cup\ A \times I \times I \end{pmatrix} \xhookrightarrow{\cong} (X \times \{0\} \cup A \times I) \times I$$

Now take the retract $X \times I \to X \times \{0\} \cup A \times I$ and multiply by I. Pull back along the above homeomorphism. This retracts $X \times I^2$ onto $X \times I \times \{0\} \cup \begin{pmatrix} X \times \{0\} \\ \cup A \times I \end{pmatrix} \times I$

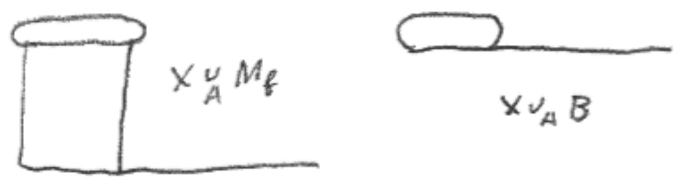
so $(X \times I, X \times \{0\} \cup A \times I)$ has HEP. Now 0.20 tells us that $X \times I$ deformation retracts onto $X \times \{0\} \cup A \times I$, so the proof of 0.18 goes through if (X_1, A) merely has HEP.

9. $X \xrightarrow{\varphi_1} X \cup_A M_f \xrightarrow{\varphi_2} X \cup_A B$

Since $A \xrightarrow{\simeq} B$, 0.21 tells us M_f def. retracts onto A, so $X \cup_A M_f$ def. retracts onto X. So φ_1 is a htpy equivalence. For φ_2 consider $X \times I \cup_{A \times \{1\}} B$:



Using the last problem, this def. retracts onto the subspaces



so

$$\begin{tikzcd} X \cup_A M_f \arrow[r, hook] \arrow[rr, bend right, "\varphi_2"'] & X \times I \cup B \arrow[r, two heads] & X \cup_A B \end{tikzcd}$$

is a htpy equivalence.