Math 215B Homework 3

Due February 1st, 2013 by 5 pm

Please remember to write down your name and Stanford ID number (9 digits). All pages and sections refer to pages and sections in Hatcher's *Algebraic Topology*.

1. (6 points) Solve §1.3 (page 79), problem 9.
2. (7 points) *Finding a covering space corresponding to a particular subgroup.* Hatcher motivates §1.3 by explicitly discussion a series of examples of covering spaces of $S^1 \vee S^1$. Read this motivation, and then solve §1.3 (page 80), problem 12.
3. (10 points) *Explicitly classifying covering spaces.* Solve §1.3 (page 80), problem 14. It may be helpful to read about Cayley graphs and complexes on pages 77-78, as a method of explicitly constructing the universal covers of some spaces—in particular, Example 1.48 gives a construction of the universal cover of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.
4. (8 points) *Covering spaces from group actions.* Given a group G and a space Y, an **action** of G on Y is a homomorphism ρ from G to $\text{Homeo}(Y)$ the group of homeomorphisms from Y to Y (multiplcation is composition). Thus, for $g \in G$, $\rho(g) : Y \to Y$ is a homeomorphism and $\rho(g \cdot g') = \rho(g) \circ \rho(g')$. For a fixed group action, ρ is often implicit, and instead we write $g : Y \to Y$ for $g \in G$. Such an action ρ is called a **covering space action** if for each $y \in Y$, there exists a neighborhood $U \subset Y$ containing y such that the collection $\{g(U) | g \in G\}$ is a disjoint collection (i.e. $g_1(U) \subset g_2(U) \neq \emptyset \Rightarrow g_1 = g_2$).
   Given any action of G on Y, one can form the **orbit space** (or quotient)
   $$Y/G := Y/\sim \text{ where } y \sim g(y) \text{ for all } g \in G. \tag{0.1}$$
   If G is a covering space action, then by Proposition 1.40 in Hatcher, then the projection $Y \to Y/G$ is a normal covering space, and moreover G is the group of deck transformations of the covering space (if Y is path-connected).
   A first example is given by the $\mathbb{Z}/2\mathbb{Z}$ action on S^n generated by the antipodal map. The quotient space is $\mathbb{RP}^n$, which we defined in class. More generally, nice group actions give us many more examples of covering spaces. (For example, take $Y = S^{2n-1}$, thought of as a subspace of $\mathbb{C}^n$, and the $\mathbb{Z}/k\mathbb{Z}$ action generated by $z \mapsto ze^{2\pi i/k}$. The orbit space is an example of a **lens space**, and has fundamental group $\mathbb{Z}/k\mathbb{Z}$ (when $n > 1$).

   Read the section in Hatcher on covering space actions (page 71-74). Then solve §1.3 (page 81), problem 23.
5. (8 points) *An example of a covering space action.* Solve §1.3 (page 81), problem 25.
6. (7 points) *Applications to group theory.* As we saw in class, covering spaces can be used to prove non-trivial facts about groups and their subgroups. In the next few exercises, we will explore some other consequences of the classification of covering spaces correspondence. It may be helpful to review §1.A and Proposition 1.32, which relates the number of sheets of a (path-connected) covering to the index of the associated subgroup.

   Solve §1.A (page 87), Problem 7.
7. (7 points) Solve §1.A (page 87), Problem 8.
8. (7 points) Solve §1.A (page 87), Problem 9.