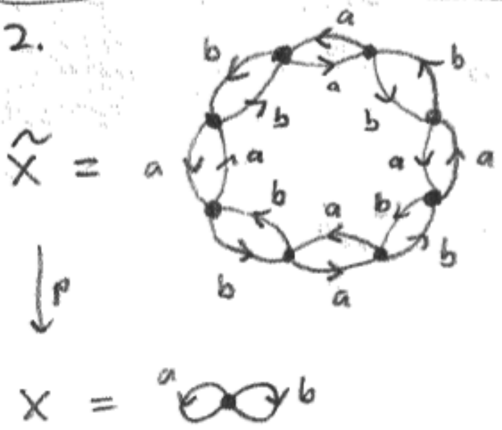
MATH 215R HW3

1. Since $\pi_1(X)$ is finite, every elt. has finite order, so $\pi_1(X) \to \pi_1(S^1)$ is 0, so a lift exists:

$$\begin{tikzcd} & \mathbb{R}^1 \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & S^1 \end{tikzcd}$$

Composing a nullhomotopy of $\tilde{f}$ with p gives a nullhomotopy of f.

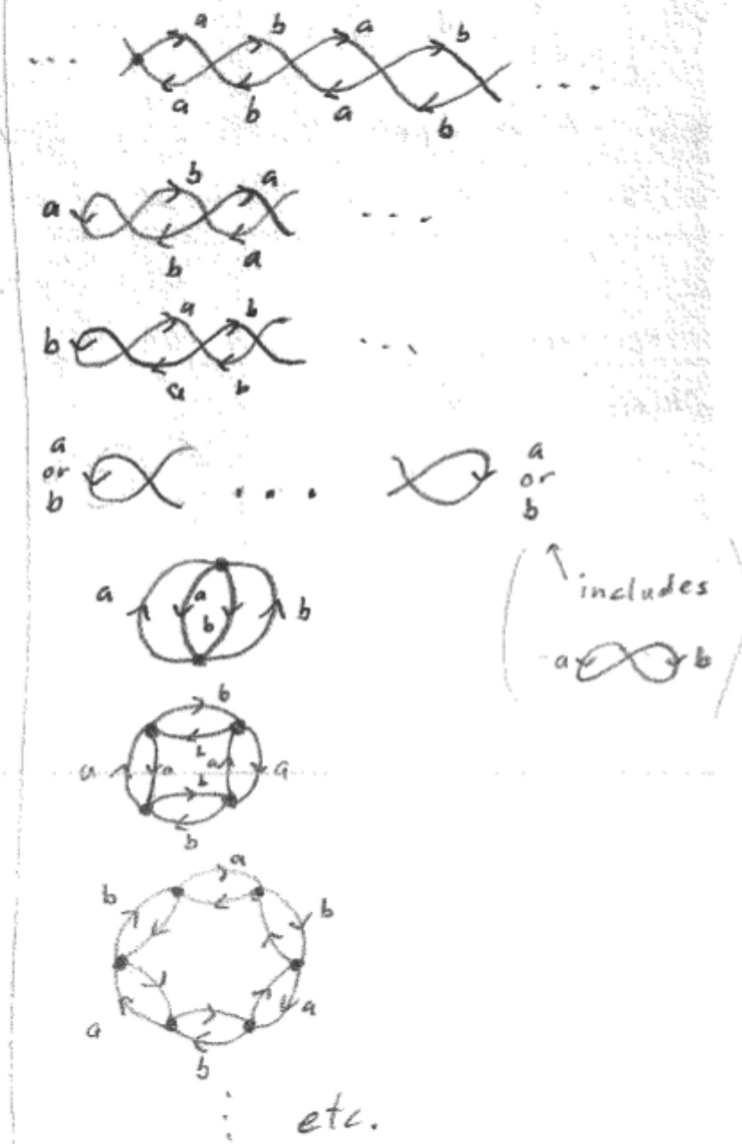
2.



This is a normal cover, so $p_* \pi_1 \tilde{X} \leq \pi_1 X$ is a normal subgroup. Let N be the smallest normal subgroup of $\pi_1 X$ containing $a^2, b^2, (ab)^4$. Then since $a^2, b^2, (ab)^4$ give closed loops upstairs, they are contained in $p_* \pi_1 \tilde{X}$, so $N \leq p_* \pi_1 \tilde{X}$. On the other hand, $\pi_1 \tilde{X}$ is free on nine generators, and we can check each one lands in N, so $p_* \pi_1 \tilde{X} \leq N$ and we are done.

3.  1-skeleton of $\mathbb{RP}^2 \vee \mathbb{RP}^2$

Each cover must have a cover of $S^1 \vee S^1$ as its 1-skeleton, and in addition, each a^2 and b^2 must be a closed loop. This results in the following possibilities:



To get the covers of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ we glue in 2-cells so that each  becomes an $\mathbb{RP}^2$ and each  becomes an S^2.

4. We have U containing x such that $\{g : g(U) \cap U \neq \emptyset\} = S$ is finite, and we want to shrink U so that this set is empty. For each $g_i \in S$, take U_i, V_i

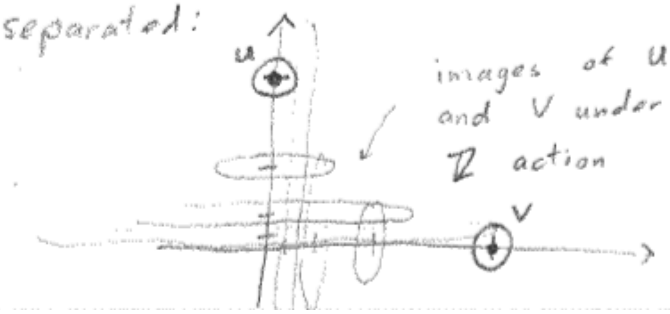
disjoint so that $x \in U_i$ and $g_i x \in V_i$. Now let

$$W = \left[\bigcap_{g_i \in S} \left(U_i \cap g_i^{-1}(V_i)\right)\right] \cap U$$

Then W is open, contains x, and all its images are disjoint, so we have a covering space action.

5. The subspace $\{(x,y) : x > 0\}$ has fundamental domain

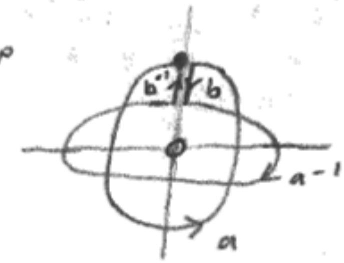
from which we easily check we have a covering space action and the quotient is $S^1 \times \mathbb{R}$. Apply this to $\{x<0\}$, $\{y>0\}$, and $\{y<0\}$ as well. Then the quotient $X/\mathbb{Z}$ is a union of four cylinders coming from these four subspaces. It is not Hausdorff since the images of $(1,0)$ and $(0,1)$ cannot be separated:



Finally, we have the short exact sequence

$$1 \longrightarrow (\pi_1 X \cong \mathbb{Z}) \hookrightarrow \pi_1 X/\mathbb{Z} \twoheadrightarrow \mathbb{Z} \longrightarrow 1$$

leaving two possibilities for $\pi_1 X/\mathbb{Z}$. To check which one we have, take a closed loop in $X/\mathbb{Z}$ representing the commutator $aba^{-1}b^{-1}$ of the two generators. Lift as a path to the cover X. It forms the loop



which is nullhomotopic. So $aba^{-1}b^{-1} = 1$ in $\pi_1 X/\mathbb{Z}$, so $\pi_1(X/\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

6. To $N \leq F$ we assign a cover of graphs $\hat{X} \longrightarrow X$ with infindely many sheets. We will show $N \cong \pi_1 \tilde{X}$ is not finitely generated. From p. 86 of Hatcher it suffices to construct infinitely many disjoint closed edgepaths in $\hat{X}$. Since $N \neq 0$, there is one such edgepath at $x_0 \in \tilde{X}$, whose length is $\ell > 0$. Since $\tilde{X}$ has infinitely many vertices, but each one has finite valence, we may inductively build a sequence of vertices $x_0, x_1, x_2, \dots$

with $d(x_i, x_j) > \ell \ \forall i,j$.
Since N is normal, our edgepath γ at x_0 may be translated to x_i for every i. These are all disjoint, so (after choice of maximal subtree) they yield an infinite subset of a set of free generators for N.

7. Let $G = F/N$, where F is free and finitely generated. Each subgroup of G of index n gives a unique subgroup of F of index n, so WLOG we may show F has finitely many such subgroups.

Fix $X = \bigvee_{i=1}^{k} S^1$ so $\pi_1 X \cong F$. Subgroups of F now correspond with based covers of X up to isomorphism. Such covers must be graphs with n vertices and nk edges. They are determined by the attaching maps of the edges, so there are at most $(n^2)^{nk}$ such graphs.

8. Pick a CW complex X with $\pi_1 X \cong G$. Then $H \leq G$ corresponds to a unique n-sheeted cover $\widetilde{X}$ of X. Let $x_0 \in X$ be the basepoint, and $\tilde{x}_1, \ldots, \tilde{x}_n \in \widetilde{X}$ its n preimages. Each conjugate gHg^{-1} is the image of $\pi_1(\widetilde{X}, \tilde{x}_i)$ for some $1 \leq i \leq n$, so there are at most n conjugates of H in G.

Conjugating by $g \in G$ simply permutes the conjugates of H, so their intersection $K \leq H$ is a normal subgroup of G. Its index is at most n^n.